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ALGEBRA

Abstract Algebra

Sagastume Berra, A. E. Modern algebra and its problems. Publ. Inst. Mat. Univ. Nac. Litoral 7, 201–218 (1946). (Spanish)

An expository lecture.

Menger, Karl. General algebra of analysis. Rep. Math. Colloquium (2) 7, 46–60 (1946). [MF 16134]

The author develops further his theory of tri-operational algebra [Algebra of Analysis, University of Notre Dame, 1944; same Rep. (2) 5–6, 3–10 (1944); these Rev. 6, 142, 143] and extends it to abstract functions of several variables. There are four sections. Section 1 is a summary of previous results, with added remarks about the representation of elements by classical functions and about the concept of fundamental polynomial.

Section 2 presents a generalization dealing with functions of rank 2, or abstract functions of two variables. Addition and multiplication are functions of two variables and may be introduced either as operations or as elements. There are now two neutral elements of substitution j_1 and j_2, such that $f(j_1, j_2)=f$, while $j_1(f, g)=f$ and $j_2(f, g)=g$. These neutral elements thus play the role of selectors. Properties of the abstract partial differentiation operators ∂_1 and ∂_2 are given.

Section 3 on functions of rank Σ deals with a further extension to functions of any number of variables, finite or infinite. This theory is only semi-algebraic. There are still no variables but set theory is used to distinguish the places at which functions are substituted into functions. In section 4 a purely algebraic theory suggested by section 3 is introduced. It concerns a ring R and a tri-operational algebra T, connected by an operation of substitution. Each element G of T may be substituted into each element f of R, giving an element fG of R. It is shown that ordinary tri-operational algebra is a special case. *O. Frink.*

Brown, Ferdinand L. The accessory postulates of tri-operational algebra. Rep. Math. Colloquium (2) 7, 61–64 (1946). [MF 16135]

The accessory postulates of tri-operational algebra are the postulates which concern the neutral elements 0, 1 and j, of addition, multiplication and substitution [cf. the preceding review for references]. The author shows that the seven accessory postulates considered by Menger may be replaced by a set of eight weaker postulates, which are shown to be independent. *O. Frink* (State College, Pa.).

Burke, John C. Remarks concerning tri-operational algebra. Rep. Math. Colloquium (2) 7, 68–72 (1946). [MF 16137]

The author defines an operator D of differentiation in polynomial tri-operational algebras in the natural way and shows that it has the usual properties of such an operator. If the constant elements form a field of characteristic p, it is shown that the operator D is single-valued only if the fundamental polynomial of the algebra is a perfect pth power. A list of examples of fundamental polynomials of degree less than 64 is given for the case $p=2$. *O. Frink.*

Mannos, Murray. Ideals in tri-operational algebras. I. Rep. Math. Colloquium (2) 7, 73–79 (1946). [MF 16138]

The author defines ideals in tri-operational algebras so that they correspond to homomorphisms onto quotient algebras. A set T of elements of a tri-operational algebra is called a T-O ideal if it is a ring ideal and if both fh and $h(f+k)-hk$ belong to T whenever f belongs to T, where fh denotes the substitution of h into f. Certain results of the theory of ideals in rings concerning maximal ideals, direct sums, and composition series are shown to carry over to T-O ideals. The commutative law of multiplication is not assumed. The T-O ideals form a modular lattice. The special case of tri-operational algebras of polynomials over a field F is considered. If F has characteristic 0, there are no proper T-O ideals. If F is a finite field, the only proper T-O ideals are principal ideals determined by a polynomial which is fundamental in a sense defined by Menger. *O. Frink.*

Habicht, Walter. Ein Existenzsatz über reelle definite Polynome. Comment. Math. Helv. 18, 331–348 (1946).

Let K be a real closed field in natural order, $f(x_1, \cdots, x_n)$ a polynomial with coefficients in K such that $f>0$ for $-m\leqq x_i\leqq m$. Then f has a positive lower bound in the same region. Because of the lack of local compactness, the usual topological methods are not available and the author gives a purely algebraic proof. A proof that f attains its lower bound is promised for a later paper. *I. Kaplansky.*

Habicht, Walter. Über die Lösbarkeit gewisser algebraischer Gleichungssysteme. Comment. Math. Helv. 18, 154–175 (1946).

According to the Poincaré-Brouwer theorem, there is no continuous field of nonvanishing tangent vectors on the 2-sphere in 3-dimensional space; the same holds for all odd dimensions. A parallel algebraic conjecture may be obtained by requiring that the direction of the tangent vectors is specified by homogeneous polynomial functions $f_1, \cdots, f_n$ of the coordinates $x_1, \cdots, x_n$, and by replacing the real field by any really closed field K. The condition of tangency is expressed by the identity $\sum x_if_i=0$. The author proves that n forms satisfying this identity have a common zero different from $(0, 0, \cdots, 0)$ in the field K, except in the case when every form f_i is of odd degree and the number of forms f_i of any given degree is even. In case K is an algebraically closed field, a similar theorem (differing only in the character of the exceptional cases) is proved. The proofs depend essentially upon an explicit formula for the general solutions of the identity $\sum x_if_i=0$ in the case when all the f_i are homogeneous and of the same degree. All solutions can be had by suitable specialization of this general solution. The existence of zeros is established by standard methods of

elimination theory [van der Waerden, Moderne Algebra, vol. 2, Springer, Berlin, 1931, chap. 11]. In the algebraically closed case, these methods make possible a determination of the general number of common zeros of the given forms. It is noted that, in case the forms f_i are linear, the results contain the theorem that a skew symmetric matrix of odd degree is necessarily singular. *S. MacLane.*

Inaba, Eizi. Über den Hilbertschen Irreduzibilitätssatz. Jap. J. Math. 19, 1–25 (1944). [MF 14991]

The Hilbert irreducibility theorem holds in a field K if, for every finite set of irreducible polynomials $f_1(x, t), \cdots, f_h(x, t)$ in two variables, one may find in K infinitely many values $t = a_i$ such that the polynomials $f_j(x, a_i)$ are irreducible as polynomials in x alone. This theorem is established in two new cases: (i) $K = k(x)$, the field of rational functions in x over a finite field k; (ii) K is a finite extension of any field L for which the theorem is known to hold. The case of a separable finite extension has been treated by W. Franz [Math. Z. 33, 275–293 (1931)]. The proofs depend on a careful analysis of the structure of the set of values of the parameter t for which the polynomials f_j remain irreducible, in the case of one or several parameters. Following the treatment of the case where K is the field of rational numbers by K. Dörge [Math. Ann. 95, 84–97 (1926)], the author considers any proper multiplicative ideal A in the Boolean algebra of all subsets of K, such that A contains all finite subsets of K. The sets of A are called thin and the Hilbert theorem holds relative to A if the allowable parameter values $t = a_i$ do not constitute a thin set. The proofs of the theorems cited above involve this thinness concept, applied to the case of several parameters, the usual Kronecker transformation of a polynomial in several variables into a polynomial in one variable, and the methods of algebraic function theory as employed by Eichler [Math. Ann. 116, 742–748 (1939)]. These methods are also used to prove the theorem in other known cases; in particular, the Hilbert irreducibility theorem is valid in any finite algebraic number field and in any field K generated from a field F by a finite number of elements, provided either that K is not algebraic over F or that the theorem is already known to hold in F. *S. MacLane.*

Whitman, Philip M. Lattices, equivalence relations, and subgroups. Bull. Amer. Math. Soc. 52, 507–522 (1946). [MF 16811]

L'auteur démontre que tout réseau $\mathfrak{L}$ est isomorphe à un sous-réseau du réseau des relations d'équivalence sur un ensemble convenable. Pour cela, il construit par un procédé qui fait intervenir l'axiome de choix un ensemble $\mathfrak{S}$ et sur cet ensemble des relations d'équivalence formant un réseau $\mathfrak{L}^*$ isomorphe a $\mathfrak{L}$. En rapprochant d'un résultat antérieur [G. Birkhoff, Proc Cambridge Philos. Soc. 31, 433–454 (1935)] il obtient: tout réseau est isomorphe à un sous-réseau du réseau des sous-groupes d'un groupe convenable. *J. Kuntzmann* (Grenoble).

Dilworth, R. P. Note on the Kurosch-Ore theorem. Bull. Amer. Math. Soc. 52, 659–663 (1946).

Les superdiviseurs de a dans un réseau modulaire sont les r tels que $r \cap x = a$ implique $x = a$. Pour que q puisse remplacer q_i dans une décomposition minimale en irréductibles de $a = q_1 \cap q_2 \cap \cdots \cap q_n = q_i \cap Q_i$ il faut et il suffit que q ne renferme pas $r \cap Q_i$ pour tout superdiviseur de a. Si on a deux décompositions minimales en irréductibles on peut numéroter les facteurs de manière que q_i' puisse remplacer q_i, mais en général pas de manière que q_i et q_i' puissent se remplacer mutuellement. Quel que soit i on pourra toujours trouver j tel que q_i et q_j' puissent se remplacer mutuellement. Accessoirement les superdiviseurs permettent de montrer l'existence d'un idéal dual consécutif à $q_1 \cap q_2 \cap \cdots \cap q_n$ sans induction transfinie. *J. Kuntzmann* (Grenoble).

Wilcox, L. R., and Smiley, M. F. Correction: Metric lattices. Ann. of Math. (2) 47, 831 (1946).

The paper appeared in the same Ann. (2) 40, 309–327 (1939).

Foster, Alfred L. Maximal idempotent sets in a ring with unit. Duke Math. J. 13, 247–258 (1946).

The author considers maximal Boolean rings in an arbitrary ring R with unit (if R is commutative, the only such subring consists of all idempotents), using $(a-b)^2$ as "sum." He shows, among other things, that in the full matrix ring over the integers mod n all maximal Boolean rings are isomorphic. *G. Birkhoff* (Cambridge, Mass.).

Ballieu, Robert. Anneaux complets de matrices rectangulaires et anneaux quasi-simples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 122–131 (1946).

The author identifies Dieudonné's quasi-simple rings with chain condition [J. Reine Angew. Math. 184, 178–192 (1942); these Rev. 5, 32] with rings of rectangular matrices. In a paper perhaps not available to the author [Bull. Soc. Math. France 70, 46–75 (1942); these Rev. 6, 144] Dieudonné proved a result [theorem 4] which includes this and extends it to rings without chain condition. *I. Kaplansky* (Chicago, Ill.).

Krasner, Marc. Théorie non-abélienne des corps de classes pour les extensions finies et séparables des corps valués complets: relations avec la théorie de la ramification; loi de limitation pour les extensions galoisiennes. C. R. Acad. Sci. Paris 222, 1370–1372 (1946).

Let $K = k(\alpha)$, let $K^{\cdot}$ be a finite algebraic extension of K and T_α the transformation $x \rightarrow f_{\alpha/k}(x)$, where $f_{\alpha/k}$ is the minimal polynomial of α over k. The skeleton [Krasner, same C. R. 219, 345–375 (1944); these Rev. 7, 363] $\mathfrak{H}_\alpha(K^{\cdot})$ of $T_\alpha K^{\cdot}$ is called the halo of $K^{\cdot}$ at α. Let $\mathcal{E}_{K/k}$ be the set of all polynomials $f_{\beta/k}(x)$ of degree $[K:k]$ such that $k(\beta) \supseteq K$. Every element γ of the skeleton $S^{\cdot}$ of $K^{\cdot}$ satisfies a unique normalized irreducible equation $f_{\gamma/s}(x) = 0$ with coefficients in s, the skeleton of k. If Q is the transformation $\gamma \rightarrow f_{\gamma/s}(x)$, then $\mathfrak{H}_f(\mathcal{E}_{K^{\cdot}/k}) = Q\mathfrak{H}_\alpha(K^{\cdot})$, where $f = f_{\alpha/k}(x)$ is called the halo of $\mathcal{E}_{K^{\cdot}/k}$ at f; $\mathfrak{H}_\alpha(K^{\cdot})$ is a retractive set [Krasner, same C. R. 222, 363–365 (1946); these Rev. 7, 429] of $S^{\cdot}$. Its minimal mutually disjoint retracts are listed and the results are then formulated in terms of $\mathfrak{H}_f(\mathcal{E}_{K^{\cdot}/k})$. *D. C. Murdoch* (Vancouver, B. C.).

Dubisch, Roy. Composition of quadratic forms. Ann. of Math. (2) 47, 510–527 (1946).

If f and g are quadratic forms in n and m variables, respectively, over a field $\mathfrak{F}$ of characteristic not 2, $f(x)$ and $g(u)$ are said to permit partial composition if there exist quantities γ_{ijk} in $\mathfrak{F}$ such that $f(x)g(u) = f(z)$, where $z_k = \sum \gamma_{ijk} x_i u_j$, $k = 1, \cdots, n$, the sum being over $i = 1, \cdots, n$; $j = 1, \cdots, m$. Conditions that forms permit partial composition when $m = n$ were found by A. A. Albert [same Ann. (2) 43, 161–177 (1942); these Rev. 3, 261]. Conditions for

$m \neq n$ are embodied in the following principal theorem of this paper.

Let $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ with a subspace $\mathfrak{M}$ and elements $x=(\xi_1, \cdots, \xi_n)$ for ξ_i in $\mathfrak{F}$, such that the partial alternative law $x(uu)=(xu)u$ holds for every x of $\mathfrak{A}$ and u of $\mathfrak{M}$. Furthermore, let $\mathfrak{A}$ possess a linear transformation J on $\mathfrak{F}$ and a unity quantity e in $\mathfrak{M}$ such that $x+x^J=et(x)$, $xx^J=ef(x)$ for every x of L, where $t(x)$ is a linear trace form and $f(x)$ is a quadratic form in the coordinates of x. Finally, let A be a nonsingular matrix such that, for $x=(\xi_1, \cdots, \xi_n)$, $f(x)=xAx'$ and let A have the property that, for every u of trace zero in $\mathfrak{M}$, the quadratic forms $(xu)Ax'$ are identically zero. Then the norm form $f(x)$ of $\mathfrak{A}$ permits the partial composition $f(x)f(u)=f(xu)$ for every x of $\mathfrak{A}$ and u of $\mathfrak{M}$. Conversely, every quadratic form permitting partial composition is equivalent to the norm form of such an algebra. More explicit results are obtained for the real field.

B. W. Jones (Ithaca, N. Y.).

Etherington, I. M. H. Corrigendum: Commutative train algebras of ranks 2 and 3. J. London Math. Soc. **20**, 238 (1945).

The writer states that the table (7.25) of the paper quoted [same J. **15**, 136–149 (1940); these Rev. **2**, 121] does not give a canonical form for every train algebra of rank 3 with train root distinct from $\frac{1}{2}$, as stated there. Other parts of the paper are said not to be affected, but the correction leaves open the question of whether every commutative train algebra of rank 3 is necessarily special.

M. H. Stone (Chicago, Ill.).

Kaplansky, Irving. On a problem of Kurosch and Jacobson. Bull. Amer. Math. Soc. **52**, 496–500 (1946). [MF 16809]

Le problème dont il s'agit est le suivant: étant donnée une algèbre A sur un corps K, dont chaque élément est racine d'un polynome à coefficients dans K, et qui admet un nombre fini de générateurs, l'algèbre A est-elle de rang fini sur K (problème analogue au célèbre problème de Burnside pour les groupes). Jacobson [Ann. of Math. (2) **46**, 695–707 (1945); ces Rev. **7**, 238] a montré que la réponse est affirmative pour les algèbres dont les éléments sont de degré non supérieur à n sur K si elle l'est pour certaines algèbres quotients bien déterminées $A(r, n)$ de l'algèbre libre à r générateurs $F(r)$ sur K. L'auteur démontre que $A(r, n)$ est bien de rang fini sur K lorsque K a au moins n éléments; il démontre aussi que $A(r, 4)$ est de rang fini lorsque K a 3 éléments; enfin, il calcule dans tous les cas le rang de $A(2, 3)$. ~~Le rapporteur signale que, dans un article qu'il n'a pu consulter [Rec. Math. [Mat. Sbornik] N.S. **13**(55), 263–286 (1943); ces Rev. **6**, 116] A. Malcev aurait également résolu le problème de Kurosch.~~

J. Dieudonné (São Paulo).

Cf. The review of a paper by Levitzki, these Rev. 8, 435.

Albert, A. A. On Jordan algebras of linear transformations. Trans. Amer. Math. Soc. **59**, 524–555 (1946). [MF 16470]

Let $\mathfrak{M}$ be the total matric algebra of all linear transformations of an n-dimensional linear space over a field $\mathfrak{F}$, not of characteristic two. A Jordan algebra is a linear subspace $\mathfrak{A}$ over $\mathfrak{F}$ of $\mathfrak{M}$ which is closed with respect to the nonassociative product $a\cdot b=(ab+ba)/2$, where ab denotes the ordinary product of the two linear transformations a, b of $\mathfrak{A}$. More generally, the author studies linear subspaces $\mathfrak{A}$ over $\mathfrak{F}$ of $\mathfrak{M}$ for which two fixed elements λ and μ not both zero of $\mathfrak{F}$ exist such that $\lambda ab+\mu ba$ belongs to $\mathfrak{A}$ for every a and b of $\mathfrak{A}$. He can show easily (in the case of an arbitrary characteristic of $\mathfrak{F}$) that $\mathfrak{A}$ then is of one of the following three types: (1) the product ab of any two elements of $\mathfrak{A}$ belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is an associative algebra; (2) the nonassociative product $a\cdot b=ab-ba$ of any two elements a, b of $\mathfrak{A}$ belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is a Lie algebra; (3) the nonassociative product $a\cdot b=(ab+ba)/2$ of any two elements a, b of $\mathfrak{A}$ belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is a Jordan algebra.

While algebras of the first two types have been widely studied, a general theory of Jordan algebras is given for the first time in the present paper. First of all, analogues of the Lie and Engel theorems on solvable Lie algebras are obtained. It is first necessary to discuss the concepts of solvability and nilpotency. If $\mathfrak{A}$ is any nonassociative algebra, the linear space of all finite sums of products $a\cdot b$ with a, b in $\mathfrak{A}$ is denoted by $\mathfrak{A}^{\cdot 2}$. Then the derived series of $\mathfrak{A}$ is defined by

$$\mathfrak{A}^{(0)}=\mathfrak{A}, \mathfrak{A}^{(1)}=\mathfrak{A}^{\cdot 2}, \mathfrak{A}^{(2)}=[\mathfrak{A}^{(1)}]^{\cdot 2}, \cdots, \mathfrak{A}^{(k)}=[\mathfrak{A}^{(k-1)}]^{\cdot 2}, \cdots.$$

If there exists an integer $k>0$ such that $\mathfrak{A}^{(k)}=0$ then $\mathfrak{A}$ is called a solvable algebra. An ideal $\mathfrak{B}$ of a nonassociative algebra is said to be solvable if $\mathfrak{B}$ is a solvable algebra (or zero). Every nonassociative algebra contains a unique maximal solvable ideal $\mathfrak{N}$. If $a_1, a_2, \cdots, a_k$ are elements of a nonassociative algebra, special products $p^{(k)}$ and ${}^{(k)}p$ are defined recursively by

$$p^{(1)}=a_1,\ p^{(2)}=p^{(1)}\cdot a_2,\ p^{(3)}=p^{(2)}\cdot a_3,\ \cdots,$$
$${}^{(1)}p=a_1,\ {}^{(2)}p=a_2\cdot {}^{(1)}p,\ {}^{(3)}p=a_3\cdot {}^{(2)}p,\ \cdots.$$

The algebra $\mathfrak{A}$ is said to be nilpotent if there exists an integer k such that the special products $p^{(k)}$ and ${}^{(k)}p$ of any k elements $a_1, a_2, \cdots, a_k$ of $\mathfrak{A}$ vanish. An algebra $\mathfrak{A}$ is strongly nilpotent if there exists an integer k such that all products of k elements of $\mathfrak{A}$ are zero. Now the theorems say that, in the case of Jordan algebras, the concepts of solvability, nilpotency and strong nilpotency are equivalent and, on the other hand, if every element of a Jordan algebra $\mathfrak{A}$ is nilpotent, then $\mathfrak{A}$ is solvable. Any subset $\mathfrak{A}$ of the total matric algebra $\mathfrak{M}$ generates an (associative) subalgebra $\mathfrak{A}^*$ of $\mathfrak{M}$; let $\mathfrak{A}^+$ denote the subalgebra of $\mathfrak{M}$ obtained by adjoining the identity I to $\mathfrak{A}^*$. Then the following remark can be added: if $\mathfrak{A}$ is a solvable Jordan algebra, then $\mathfrak{A}^*$ is the radical of the associative algebra $\mathfrak{A}^+$. The radical $\mathfrak{N}$ of an arbitrary Jordan algebra $\mathfrak{A}$ is defined as the maximal solvable ideal of $\mathfrak{A}$. Then $\mathfrak{N}$ is the intersection of $\mathfrak{A}$ with the radical of the associative algebra $\mathfrak{A}^*$.

An important tool for building up the structure theory of associative algebras is formed by the Peirce decomposition. It is therefore important that an analogue of the decomposition can be obtained. Using this, the author develops the structure theory. It is now assumed that $\mathfrak{F}$ is a nonmodular field. In this case, the radical $\mathfrak{N}$ of a Jordan algebra $\mathfrak{A}$ can be characterized by a trace condition. In fact, $\mathfrak{N}$ consists of those elements q of $\mathfrak{A}$ for which the trace of $a\cdot q$ vanishes for all a in $\mathfrak{A}$. A Jordan algebra is semisimple if its radical vanishes. Every semisimple Jordan algebra is a direct sum of simple algebras. Every semisimple algebra contains a unity element e such that $ea=ae=e\cdot a=a$. If the Jordan algebra $\mathfrak{A}$ is not semisimple, then the residue class algebra $\mathfrak{A}$–$\mathfrak{N}$ of $\mathfrak{A}$ modulo its radical $\mathfrak{N}$ is semisimple, and this shows that the above definition of the radical is consistent with the definition of the radical of a nonassociative algebra given by the author in an earlier paper [Bull. Amer. Math. Soc. **48**, 891–897 (1942); these Rev. **4**, 130].

In the last chapter, simple Jordan algebras are studied. The center $\mathfrak{C}$ of a Jordan algebra $\mathfrak{A}$ is the set of all elements c of $\mathfrak{A}$ such that the associative products ac and ca are

equal. Then $ac=a\cdot c$. The center $\mathfrak{C}$ of a simple Jordan algebra is a field. A simple Jordan algebra $\mathfrak{A}$ with the unity element e is said to be central simple if the center of $\mathfrak{A}$ is the field $e\mathfrak{F}$ isomorphic to $\mathfrak{F}$. Every simple Jordan algebra is expressible as a central simple Jordan algebra over its center. Every scalar extension of a central simple Jordan algebra is again central simple. As in the ordinary theory, an idempotent u of a Jordan algebra is said to be primitive if u cannot be written as the sum of two pairwise orthogonal idempotents of $\mathfrak{A}$. A reduced algebra is defined as a simple algebra such that $u\mathfrak{A}u$ has order one for every primitive idempotent u of $\mathfrak{A}$. The author succeeds in a complete construction of all reduced Jordan algebras. On the other hand, if $\mathfrak{A}$ is an arbitrary central simple Jordan algebra, there exist extension fields $\mathfrak{K}$ of $\mathfrak{F}$ of finite degree such that the corresponding scalar extension of $\mathfrak{A}$ is reduced. If scalar extensions of $\mathfrak{F}$ are admitted, we may therefore say that all central simple Jordan algebras are obtained. The rôle of the total matric algebra in the theory of associative algebras is played in the theory of Jordan algebras by what the author calls split algebras. There are four types of such algebras: (1) the Jordan algebra of all square matrices of degree t; (2) the Jordan algebra of all symmetric matrices of degree t; (3) the Jordan algebra of all matrices a of degree $2t$ for which $a=Pa'P^{-1}$, where P is the direct sum of t matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

(4) the algebras with a basis $1, u_2, \cdots, u_{s+2}$, where $u_i^2=1$, $u_i\cdot u_j=0$ for $i\neq j$ and $s>4$ or $s=3$.

If $\mathfrak{A}$ is a central simple algebra over a nonmodular field $\mathfrak{F}$, there exist extension fields $\mathfrak{K}$ of finite degree such that the corresponding extensions of $\mathfrak{A}$ are split algebras. These fields $\mathfrak{K}$ then are called the splitting fields of $\mathfrak{A}$. An invariant t, the degree of the central simple algebra, is defined which plays the same rôle as the degree of a central simple associative algebra. Each split algebra is uniquely determined by its degree and order. *R. Brauer* (Toronto, Ont.).

Nesbitt, C. J., and Thrall, R. M. Some ring theorems with applications to modular representations. Ann. of Math. (2) 47, 551–567 (1946).

The paper consists of three parts with contents as follows. (1) Let I be an integral domain in an algebra A over a π-adic field, W a representation of I, and W^* the induced representation of $I^*=I$ mod π. Then any block constituent of W^* is likewise an induced representation. For group rings, this result had been announced earlier by the second author [Bull. Amer. Math. Soc. 50, 335 (1944)]. (2) If V is a faithful representation of a quasi-Frobenius algebra, then V is the commutator of its commutator. [For A semisimple this is proved in Artin, Nesbitt and Thrall's "Rings with Minimum Condition," Univ. Michigan Publ. in Math., no. 1, 1944, p. 42; these Rev. 6, 33]. (3) Orthogonality relations for Frobenius algebras are developed in two different ways. *I. Kaplansky* (Chicago, Ill.).

Yamada, Kaneo. Liesche Ringe und assoziative hyperkomplexe Systeme. Tôhoku Math. J. 48, 152–166 (1941). [MF 16355]

The author discusses the isomorphic embeddings of Lie algebras in associative algebras A, such that L generates A (that is, A is an "enveloping algebra" of L). Only the case of fields of characteristic infinity and algebras with finite basis is discussed. He shows that if $L=L^*+R$ is the E. Levi-J. H. C. Whitehead decomposition of L into its greatest solvable invariant subalgebra ("radical") R and a semisimple subalgebra L^*, then any enveloping algebra A has a Wedderburn decomposition $A=A^*+N$ into its radical N and a semisimple subalgebra A^*, such that $A^*\geqq L^*$. Again, any enveloping algebra A of a semisimple Lie algebra L is semisimple, and direct decompositions of A induce corresponding direct decompositions of L into direct summands, but not conversely. If A is normal simple, direct decompositions of L into direct summands induce representations of A as a direct product. All semisimple Lie rings having a fixed simple A for enveloping algebra form a semilattice. The author concludes by trying to establish a one-to-one correspondence between normal simple Lie rings and certain associative algebras. *G. Birkhoff* (Cambridge, Mass.).

Hochschild, G. On the cohomology theory for associative algebras. Ann. of Math. (2) 47, 568–579 (1946).

In this paper the author continues his investigation of the cohomology groups of algebras which he introduced in an earlier paper [Ann. of Math. (2) 46, 58–67 (1945); these Rev. 6, 114]. He shows that the cohomology group $H^n(A, P)$ of the algebra A in the A-module P is essentially unchanged if a (possibly missing) identity is adjoined to A; if 1 is the identity in A, then $H^n(A,P)\cong H^n(A, 1\cdot P)\cong H^n(A, P\cdot 1)\cong H^n(A, 1\cdot P\cdot 1)$, so that it does not make any difference whether or not the identity in A acts as an identity in P. Next he shows that $H^n(A\oplus B, P)\cong H^n(A,P)\oplus H^n(B,P)$, provided the algebras A and B possess identities. Finally, he proves that $H^2(A,P)\neq 0$ for at least one P if A is either nilpotent or semisimple but inseparable. There exist, however, inseparable algebras A such that $H^2(A,P)=0$ for every P. *R. Baer* (Urbana, Ill.).

ANALYSIS

Ostrowski, A. Sur l'intégrabilité élémentaire de quelques classes d'expressions. Comment. Math. Helv. 18, 283–308 (1946).

A method is furnished for deciding the possibility of integrating in finite terms any rational combination of log z and z. This method is extended as follows. Let R be any differential field of meromorphic functions. Let w be an integral of a function in R, it being understood that w is not obtainable from functions of R by performing algebraic, logarithmic and exponential operations repeated any finite number of times. It is shown how to test for integrability in finite terms any rational expression in w with coefficients in R. *J. F. Ritt* (New York, N. Y.).

Ostrowski, Alexandre. Sur les relations algébriques entre les intégrales indéfinies. Acta Math. 78, 315–318 (1946).

Let R be a differential field of meromorphic functions which contains all constants. Let $\varphi_1, \cdots, \varphi_n$ be n functions in R. For each i, let ψ_i be some indefinite integral of φ_i. Suppose that an algebraic relation exists among the ψ_i, the coefficients in the relation being functions in R. It is proved that there exists a relation $a_1\psi_1+\cdots+a_n\psi_n=\gamma$, where the a_i are constants and γ belongs to R. *J. F. Ritt.*

Seebach, Karl. Über die Erweiterung des Definitionsbereiches mehrmals differenzierbarer Funktionen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 67–90 (1941).

Let M be a closed subset of the x-axis and let M' be the derived set. Let $f(x)=f^{(0)}(x), \cdots, f^{(k)}(x)$ be defined in M such that $f^{(\nu-1)}(x)=f^{(\nu-1)}(x_0)+(x-x_0)f^{(\nu)}(x_0)+o(x-x_0)$ for $\nu=1, \cdots, k$, $x\epsilon M$, $x_0\epsilon M'$. A necessary and sufficient condition is given for the existence of a function $F(x)$ defined for all x whose νth derivative equals $f^{(\nu)}(x)$ in M. Given $F(x)$, define, in each complementary interval I_n to M, the function $h(x)$ as the difference between $F(x)$ and the linear function with the value of F at the ends of the interval. The condition is that there exists such a function $h(x)$ with five given properties. [For related work, see references in papers by J. de Groot, Mathematica, Zutphen. B. **12**, 15–24 (1943); these Rev. **7**, 277, and the reviewer, Bull. Amer. Math. Soc. **50**, 76–81 (1944); these Rev. **5**, 202.]

H. Whitney (Cambridge, Mass.).

Rodov, A. Relations between upper bounds of derivatives of functions of a real variable. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 257–270 (1946). (Russian. English summary)

Let $f(x)$ be bounded and have n bounded derivatives on $(-\infty, \infty)$; let M_k be the least upper bound of $|f^{(k)}(x)|$. The problem of necessary and sufficient conditions on the numbers a_0, a_k, a_n so that there is a function $f(z)$ with $M_j=a_j$, $j=0, k, n$, was solved by Kolmogoroff [C. R. Acad. Sci. Paris **207**, 764–765 (1938); Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 3–16 (1939); these Rev. **1**, 298]. The author now sets the corresponding problem for $M_j=a_j$, $j=0, i_1, \cdots, i_k, n$. He solves it for $M_j=a_j$, $j=0, n-2, n-1, n$ and $j=0, k, n-2, n-1, n$ $(0<k<n-2)$. In particular, he obtains sharp inequalities for M_0, M_1, M_2, M_3; for M_0, M_1, M_2, M_3, M_4; and for M_0, M_2, M_3, M_4. For M_0, M_1, M_2, M_3 his result is as follows: $M_1\geqq\frac{1}{2}M_2/M_3$, $M_0\geqq\Psi(M_1, M_2, M_3)$, where Ψ is the maximum absolute value of $\psi(x)$, defined as follows: let $l_1=M_1M_2^{-1}-\frac{1}{2}M_2M_3^{-1}$, $l=M_1M_2^{-1}+\frac{1}{2}M_2M_3^{-1}$; then $\psi'''(x)=M_3$ in $l_1<x<2l-l_1$, $\psi'''(x)=-M_3$ in $2l+l_1<x<4l-l_1$, $\psi'''(x)=0$ elsewhere in $(0, 4l)$; $\psi''(l)=0$, $\psi'(2l)=0$, $\psi(3l)=0$; $\psi(x)$ has period $4l$ (then $\Psi(M_1, M_2, M_3)=\psi(4l)$). For the other cases the results are expressed in terms of similar but more complicated functions. The idea of the proof is to compare the given $f(x)$ with a function which is the nth integral of a step-function and has the same M_k's as $f(x)$ for the prescribed values of $k>0$. For example, in the case of M_0, M_1, M_2, M_3, the function $\psi(x)$ described above has $\max|\psi'''|=M_3$, $\max|\psi''|=M_2$, $\max|\psi'|=M_1$; it can then be shown that if $|f'(3l)|=M_1$, the assumption $|f(4l)|<\psi(4l)$ leads to a contradiction. [Not all the statements in the author's summary correspond to the Russian text.]

R. P. Boas, Jr. (Providence, R. I.).

Obrechkoff, Nikola. Sur quelques inégalités pour les dérivées des fonctions d'une variable réelle et pour les différences des suites. C. R. Acad. Sci. Paris **223**, 397–399 (1946).

The author gives further consequences of the inequalities announced in an earlier note [same C. R. **222**, 531–533 (1946); these Rev. **7**, 419]. He also gives an analogous inequality for differences of a numerical sequence. One of the theorems is the following result on solutions of the differential equation $y^{(n)}=f(x, y)$, where $f(x, y)$ is continuous and satisfies $|f(x, y)|\leqq\Phi(x)|y|$, the integral $\int^\infty t^{n-1}\Phi(t)dt$ being convergent: every solution $y(x)$ which satisfies $\limsup_{x\to\infty} x^{-m}|y(x)|<\infty$, m an integer less than n, is nonoscillating as $x\to\infty$. *R. P. Boas, Jr.* (Providence, R. I.).

Bang, Thøger. A moment problem. Mat. Tidsskr. B. **1946**, 77–82 (1946). (Danish) [MF 16308]

The author considers the compatibility of the following conditions: if u_r are a finite number of points and $0\leqq j_r<n$, then $f^{(j_r)}(u_r)=a_r$. He supposes that $f(x)=(\int^x dx)^n f^{(n)}(x)$. Then to every $f^{(j_r)}(u_r)$ corresponds a $K_r(x)$ such that $f^{(j_r)}(u_r)=\int_{-\infty}^{\infty}f^{(n)}(x)K_r(x)dx$. He proves that the conditions are compatible if and only if, for any sequence $\{c_r\}$, $\int|\sum c_rK_r(x)|dx-\sum c_ra_r\geqq0$. *František Wolf.*

Rosenbloom, P. C. Some properties of absolutely monotonic functions. Bull. Amer. Math. Soc. **52**, 458–462 (1946). [MF 16801]

The paper contains several fragmentary properties of absolutely monotonic functions of which we state here only the first. If $f(x)$ is absolutely monotonic in the range $0\leqq x<a$ and $L(x)$ is the Lagrange interpolation polynomial of $f(x)$ at the points $x_1, \cdots, x_n$, then

$$g(x)=\{f(x)-L(x)\}/\{(x-x_1)(x-x_2)\cdots(x-x_n)\}$$

is an absolutely monotonic function of the $n+1$ variables $x, x_1, \cdots, x_n$ in the range $0<x, x_1, \cdots, x_n<a$.

I. J. Schoenberg (Philadelphia, Pa.).

Vicente Gonçalves, J. Le théorème de M. S. Bernstein. Portugaliae Math. **5**, 135–136 (1946).

A simple proof of the theorem that $f(x)$ is analytic in (a, b) if all its derivatives are nonnegative there; a somewhat stronger result is also established.

R. P. Boas, Jr. (Providence, R. I.).

Hardy, G. H. Riemann's form of Taylor's series. J. London Math. Soc. **20**, 48–57 (1945). [MF 15934]

Riemann's form of Taylor's series is

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}D^{m+r}f(x), \tag{1}$$

where the author assumes that $0<r<1$, $h>0$, and the operaton of generalized differentiation is defined by

$$\begin{aligned} D^pf(x)&=\frac{1}{\Gamma(q)}\int_a^x(x-t)^{q-1}f(t)dt, \quad q=-p>0,\\ D^{m+p}f(x)&=(d/dx)^mD^pf(x), \quad p<0;\ m=1, 2, 3, \cdots, \end{aligned} \tag{2}$$

where a is a suitable constant. The discussion is restricted to functions $f(x)$ of two special types. In the first case $f(x)$ is an integral function and $f(x)=O(e^{c\Re x})$ $(c>0)$ for $\Re x<0$. In this case $a=-\infty$ in (2) and $x>0$ in (1). In the second case $f(x)$ is regular for $\Re x>0$, $f(x)=O(|x|^n)$ $(n>0)$ for x small and $\Re x>0$. In this case $a=0$ in (2) and $0<h<x$ in (1). Interesting special functions of these types are e^{cx} and x^n, respectively. In either case the series (1) is decomposed as follows: $\sum_{-\infty}^{-1}+\sum_0^\infty=S_1+S_2$; it is then shown that S_2 converges in the ordinary sense. The series S_1 is summed by the following generalized Borel method. If the series $a(w)=\sum_0^\infty a_\nu w^\nu/\nu!$ converges for small w and $a(w)$ is regular for $w>0$ with

$$\int_0^\infty e^{-w}a(w)dw=s,$$

then s is the (B^*)-sum of $\sum a_\nu$. In this sense the formula (1) is shown to hold in both cases. *I. J. Schoenberg.*

***Marke, Poul W.** **Bidrag til Teorien for Integration og Differentiation af vilkaarlig Orden.** **[Contribution to the Theory of Integration and Differentiation of Arbitrary Order].** Thesis, University of Copenhagen, 1942. 127 pp. (Danish)

The first chapter discusses fractional integrals and derivatives with a finite origin. The principal new result connects them with fractional differences: for an integrable $f(x)$, its difference quotients of negative order k converge in mean to its $(-k)$th integral; its difference quotients of positive order k converge in mean if and only if it has an integrable kth derivative.

The second chapter discusses integrals and derivatives with origin $-\infty$. The author makes use of the integral and derivative "over a length L," that is, for example, for the kth integral,

$$_{x-L}J_x^k f(x) = \Gamma(k)^{-1} \int_0^L f(x-t) t^{k-1} dt,$$

observing that the integral with origin $-\infty$ can be defined as $\lim_{L\to\infty} {}_{x-L}J_x^k f(x)$, and similarly for derivatives. Combinations of integrations and differentiations are discussed in great detail, conditions being given both for the expression of two successive operations on a function as a single one and for the decomposition of one operation into the product of two. The principal result on composition of more than two operations is that to each $f(x)$ there is a polynomial $P(x)$ such that any sequence of integrations and differentiations applied to $f(x)$ can be reduced to (a) a differentiation of nonnegative order or (b) one or more integrations of positive order, applied to $f(x)-P(x)$.

In the third chapter the results are applied to the class of functions with a finite S-norm $D_S[f(x)]$, suggested by Stepanoff's definition of almost periodicity. Let

$$D_S[f(x), g(x)] = \sup_{-\infty<x<\infty} \int_{x-1}^{x} |f(t)-g(t)|\,dt,$$
$$D_S[f(x)] = D_S[f(x), 0].$$

The S-Lipschitz class S-Lip α consists of functions such that $D_S[f(x-\delta), f(x)] < c\delta^\alpha$, $0<\delta\leqq 1$. Theorems are given on the S-Lip class of integrals and derivatives of S-bounded functions. Results are then given on the S-almost periodicity of integrals and derivatives of S-almost periodic functions.

Among the results announced without proof at the end are some on k-fold monotone functions where k is not necessarily integral, in particular, a generalization of Grüss's inequality [see Hardy, J. London Math. Soc. 11, 167–170 (1936)]. *R. P. Boas, Jr.* (Providence, R. I.).

Harmonic Functions, Potential Theory

Kappos, D. A. **Das Dirichletsche Problem für Gebiete mit mehrfachen Randpunkten.** Ann. Scuola Norm. Super. Pisa (2) 11, 44–63 (1942). [MF 16752]

This paper contains a generalization of a part of the theory developed by the reviewer in connection with the Dirichlet problem for domains with multiple boundary points [Trans. Amer. Math. Soc. 38, 106–144 (1935)]. The discussion is formulated for space of three dimensions; it is stated that corresponding results are valid in the plane. The discussion is limited to a domain G with a finite boundary, although G itself may be infinite. The author interprets the interior points of G and the boundary elements as points of a metric space and shows that if $f(\gamma)$ is a metrically continuous function of the boundary elements then a necessary and sufficient condition that there exist a function u harmonic in G which assumes the boundary values $f(\gamma)$ with metric continuity is that G be normal in the classical sense. The corresponding theorem in the earlier paper gives a similar sufficient condition for the case in which G is bounded and $f(\gamma)$ is uniformly pseudo-continuous; under the same restrictions the corresponding necessary condition is readily established. The author makes use of a theorem due to Tietze [J. Reine Angew. Math. 145, 9–14 (1914)] on the metrically continuous extension of functions which is more general than the corresponding theorem in the reviewer's paper.

It may be noted that some aspects of the Dirichlet problem for domains with multiple boundary points have already been studied from the point of view of the theory of metric spaces by J. W. Green [Amer. J. Math. 61, 609–632 (1939); these Rev. 1, 17]. *F. W. Perkins* (Hanover, N. H.).

Sherman, D. **On the reduction of the plane problem of the theory of potential to an integral equation.** Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 357–362 (1945). (Russian. English summary)

Let S be a finite plane domain whose boundary L consists of a finite number of closed curves of continuous curvature. The author considers the problem of determining a function $u(x, y)$ harmonic in S and satisfying on L a differential condition of the form $a(s)\partial u/\partial x + b(s)\partial u/\partial y + c(s)u = f(s)$, or more generally of the form

$$\sum_{k=0}^{m} \sum_{j=0}^{k} a_{kj}(s) \frac{\partial^k u}{\partial x^{k-j} \partial y^j} = f(s),$$

where $a_{kj}(s)$ and $f(s)$ are suitably restricted functions of the arc length; he reduces the problem to the solution of a Fredholm equation. *E. F. Beckenbach.*

Sherman, D. **On some problems of the theory of stationary oscillations.** Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 363–370 (1945). (Russian. English summary)

The method of the paper reviewed above is extended to treat similarly the problem of determining solutions of the differential equation $\Delta u + \lambda^2 u = 0$, with boundary condition the same as in the former problem. *E. F. Beckenbach.*

Scherman, D. J. **On Poincaré's problem in the theory of potential.** C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 636–639 (1945).

On considère dans le plan un domaine fini multiplement connexe limité par $m+1$ courbes de Jordan à courbure continue et ne se coupant pas. Le problème de Poincaré consiste à y trouver une fonction harmonique u telle que u, $\partial u/\partial x$, $\partial u/\partial y$ satisfassent à une relation linéaire donnée sur la frontière. L'auteur donne de u une expression intégrale portant sur une fonction $\nu(s)$ de l'arc, où $\nu(s)$ est solution d'une équation de Fredholm, dont l'équation associée correspond aussi à un problème de Poincaré. *M. Brelot.*

Sherman, D. **On the general problem of the potential theory.** Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 121–134 (1946). (Russian. English summary)

On donne dans le plan $z=x+iy$ un domaine fini limité par une ou $p+1$ courbes de Jordan à courbure continue.

On y cherche une fonction harmonique u satisfaisant sur la frontière à la condition

$$\sum_{k=0}^{m} \sum_{j=0}^{k} a_{kj}(s) \frac{\partial^k u}{\partial x^{k-j} \partial y^j} = f(s).$$

L'entier m peut varier avec chaque contour partiel; f est continue; les a_{kj} satisfont à la condition de Lipschitz et à $|\sum_{j=1}^{m} (i)^j a_{mj}| \neq 0$. Le problème est ramené à une équation de Fredholm. *M. Brelot* (Grenoble).

Monna, A. F. Sur la représentation des fonctions harmoniques. Nederl. Akad. Wetensch., Proc. 44, 939–942 (1941). [MF 15760]

In an earlier paper [same Proc. 44, 718–726 (1941); these Rev. 3, 47] the author considered the representation of a positive harmonic function u in a domain Ω with boundary Σ by a Stieltjes integral, $u(P) = \int_\Sigma \gamma(P, Q) d\theta(e_Q)$, where $\theta(e)$ is a positive mass distribution depending on u and $\gamma(P, Q)$ depends only on Ω. Actually $\gamma(P, Q) = \lim \mu(P, e)/\mu(P_0, e)$, where $\mu(P, e)$ is harmonic measure at P of a set e on Σ, P_0 is any fixed point in Ω and the limit is taken over a suitable sequence of sets e tending to Q on Σ. The proof of the representation was sketchy and the question of what conditions must be imposed on Ω was not discussed; it is this question that is the concern of the present paper. Let Ω_n be a sequence of domains in Ω tending to Ω, let Σ_n be the boundary of Ω_n and let $Q_n \subset \Sigma_n$ be a sequence of points tending to Q on Σ. The sufficient condition obtained amounts to the fact that, when one considers the function $\gamma(P, Q)$ for the approximating domains Ω_n, $\gamma_n(P, Q_n) = \lim \mu_n(P, e)/\mu_n(P_0, e)$, $e \to Q_n$, $e \subset \Sigma_n$, the limit is uniformly approached for all n. As is customarily the case for such sufficient conditions, it is not subject to easy geometrical interpretation. *J. W. Green.*

Monna, A. F. Sur un principe de variation de Gauss dans la théorie du potentiel. Nederl. Akad. Wetensch., Proc. 49, 54–62 = Indagationes Math. 8, 18–26 (1946). [MF 16562]

L'auteur reprend son étude antérieure [mêmes Proc. 44, 50–61 (1941); ces Rev. 2, 293]. Soit dans l'espace un domaine τ, de frontière Σ bornée et tout entière extérieure. Si θ est une distribution de masses positives dans τ, de potentiel V, l'auteur étudie les distributions positives ρ sur Σ, de potentiel U satisfaisant à $U \leqq V$ sur $\tau \cup \Sigma$, $U = V$ sur le complémentaire. Il montre à nouveau en particulier que $\int (U - 2V) d\rho$ est maximum seulement quand ρ est la distribution obtenue par extrémisation de θ sur $\tau \cup \Sigma$. *M. Brelot* (Grenoble).

Rydbeck, Olof E. H. On the spherical and spheroidal wave functions. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] 1945, no. 43, 34 pp. (1945). [MF 16820]

The author starts by a discussion of spherical and hyperspherical wave functions, introducing hyperpolar coordinates into the Laplace equation for n independent variables. A hyperspherical surface harmonic of degree m_1 is expanded in an infinite product of angular harmonic functions, generalizations of Gegenbauer potential coefficients. Specializations in the case of two, three and four dimensions are given. Using Hobson's derivation theorem, a theorem on the expansion of Gegenbauer derivatives of the fundamental radial hyperspherical function is obtained, of which some special cases are shown to coincide with well-known relations. Integrating a plane wave over an n-dimensional solid angle, a definite integral expression for $J_{\nu+m_1}(kr)/(kr)^\nu$ is established and the integrand is identified with the Fourier integral frequency spectrum of the function. A physical low-pass filter interpretation of the cut-off frequency in the spectrum is given. Starting from a definite integral expression for the fundamental radial hyperspherical function a number of further Fourier integral expansions, for example, for Gegenbauer functions of the second kind, are obtained. Finally, an addition theorem for radial hyperspherical functions is established. A similar discussion is extended to the case of prolate hyperspheroidal coordinates. Expanding the solution of the radial hyperspheroidal potential equation in a series of general Bessel-Hankel functions, the coefficients are obtained from recurrent relations, the ratio of two consecutive coefficients being determined by a continued fraction. Similarly the angular hyperspheroidal functions are expanded in a series of Gegenbauer functions, again with continued fractions for the ratios of two consecutive coefficients. In the case of two dimensions well-known Mathieu function formulas follow from this discussion. Relationships are obtained between the coefficients of the radial and the angular functions. A number of further relations between the functions under discussion, including addition formulas, are set forth. *M. J. O. Strutt* (Eindhoven).

Vekoua, I. N. Représentation générale des solutions d'une équation différentielle des fonctions sphériques. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 311–314 (1945).

This note contains various representations of spherical harmonics $u(\theta, \varphi)$ of degree n. In suitable complex variables z, ζ the differential equation of these functions takes the form

$$\frac{\partial^2 u}{\partial z \partial \zeta} + \frac{n(n+1)u}{(1+z\zeta)^2} = 0.$$

A typical example of the representations given for the most general solution of this equation, which is regular in a neighborhood of the point $\theta = 0$, is the following;

$$u = \varphi(z) - \int_0^z \varphi(t) \frac{\partial}{\partial t} P_n \left\{ \frac{1 - z\zeta + 2\zeta t}{1 + z\zeta} \right\} dt + \psi(\zeta) - \int_0^\zeta \psi(\xi) \frac{\partial}{\partial \xi} P_n \left\{ \frac{1 - z\zeta + 2z\xi}{1 + z\zeta} \right\} d\xi,$$

where φ and ψ are arbitrary analytic functions and P_n denotes the Legendre function of the first kind of degree n. *F. John* (New York, N. Y.).

Vecoua, I. N. Integration of equations of a spherical shell. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 368–388 (1945). (Russian. English summary) [MF 15695]

This paper presents an expression for all solutions of the equation

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} + n(n+1) u = 0,$$

where θ and φ are spherical coordinates on a unit sphere. A number of applications are made of the general solutions, the most important one being that to the theory of thin shells. Making use of a result of Goldenweiser [same journal 8, 441–467 (1944); these Rev. 7, 42] the author derives the general expression for the stresses, moments and components of displacements of thin spherical shells in terms of four arbitrary analytic functions of a complex variable. *H. P. Thielman* (Ames, Iowa).

Charczenko, Pierre. Note sur certaines formules de géophysique employées dans la prospection gravimétrique. Ann. Géophysique 2, 93–96 (1946).

Direct computation of a well-known expression for the derivatives of the gravitational potential caused by a horizontal half-plane of finite thickness (horizontal layer). The result is not new and the computation could be much simplified using the expression of the potential as a complex integral. *E. Kogbetliantz* (New York, N. Y.).

Soudan, R. Sur les polydromies des fonctions biharmoniques. Comment. Math. Helv. 18, 52–58 (1945).

By a generalization of Green's formula (Gutzmer's formula) and by an extension of the Cauchy-Kowalewski theorem the author establishes a method of extending to polyharmonic functions in n-dimensional space the work of R. Wavre [Prace Mat.-Fiz. 44, 75–89 (1937); Math. Z. 37, 739–748 (1933)]. *A. Gelbart* (Syracuse, N. Y.).

Soudan, Robert. Sur les polydromies des fonctions biharmoniques. C. R. Séances Soc. Phys. Hist. Nat. Genève 62, 49–51 (1945). [MF 14556]

A summary of the results of the paper reviewed above. *A. Gelbart* (Syracuse, N. Y.).

Kim, [Yu. C.]. Die Randwertaufgabe für polyharmonische Funktionen. Bull. Soc. Phys.-Math. Kazan (3) 12, 147–170 (1940). (Russian. German summary) [MF 13905]

Using the method of Muschelišvili and results of Sherman [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 2, 163–225 (1937)], the author gives a new solution of the boundary value problem for the polyharmonic equation. He determines the solution U of the equation $\Delta^n U=0$, regular in the domain S, and satisfying boundary conditions $U=F_1(s)$, $\partial U/\partial n=F_2(s)$, $\Delta U=F_3(s)$, $\partial\Delta U/\partial n=F_4(s)$, $\cdots$, $\Delta^{(n-1)/2}U=F_n(s)$ or $(\partial\Delta^{(n-2)/2}U/\partial n)=F_n(s)$. Here n denotes the interior normal. The author proves first that U can be represented in the form

$$U=\tfrac{1}{2}\sum_{\nu=1}^{n}\{\bar{z}^{(\nu-1)}\varphi_\nu(z)+z^{(\nu-1)}\bar{\varphi}_\nu(\bar{z})\}$$

and then derives the integral equation of Fredholm type which U has to satisfy. The existence and uniqueness of the solution are proved. *S. Bergman* (Cambridge, Mass.).

Khalilov, Z. I. Le problème limite général pour un système d'équations polyharmoniques généralisées. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 171–173 (1946).

Il s'agit de systèmes de m équations à m fonctions inconnues de deux variables indépendantes, avec conditions aux limites linéaires, de la forme dite "polyharmonique." Ce problème, dont un cas particulier a été étudié par Gevrey [C. R. Acad. Sci. Paris 171, 610–612, 839–842 (1920); 173, 761–763, 1445–1447 (1921)], généralise certains questions posées par la théorie de la déformation ou de l'oscillation des systèmes élastiques à deux dimensions. Utilisant des méthodes indiquées par I. Vekua [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 109–139 (1942); ces Rev. 6, 123], l'auteur le ramène à la recherche d'un système de n fonctions holomorphes satisfaisant certaines conditions aux limites, et, en utilisant une représentation "intégrale" de ces fonctions, à un système d'équations intégrales singulières, auquel, dans le cas normal, s'applique une théorie indiquée par Muschelišvili et N. P. Vekua [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 12, 1–46 (1943); ces Rev. 6, 272]. *M. Janet* (Paris).

Differential Equations

***Lefschetz, Solomon. Lectures on Differential Equations.** Annals of Mathematics Studies, no. 14. Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1946. viii+210 pp. $3.00.

The author has prepared this monograph in order "to provide the necessary background and preparation" for those interested in differential equations in the large. The reader will find the fundamental facts about systems of differential equations along with basic and useful material on stability based on the work of Liapounoff and Poincaré. This latter work is not contained in any contemporary treatment. Moreover, by using the notions of matrices and vector spaces, along with simple topological notions, the author achieves great terseness as well as clarity.

The first chapter takes up introductory material on matrices, vector spaces and analytic functions of several variables. Chapter II contains existence theorems and theorems on continuity and analyticity including the case of systems involving parameters. Chapter III is on homogeneous and nonhomogeneous linear systems. Chapter IV is on stability. Chapter V is on the results of Poincaré and Bendixon for two-dimensional systems. The results obtained in chapter IV on behavior near critical points are sufficiently precise to allow many results in chapter V on two dimensional systems to be proved simply as special cases of theorems in chapter IV. In chapter VI an application of the Poincaré method of small parameters is made to the van der Pol equation. The existence of at least one periodic solution for certain second order systems with periodic forcing terms is proved.

[The reference to Liénard's paper for the proof of the existence of a unique periodic solution for the equation $\ddot{x}+f(x)\dot{x}+g(x)=0$ is incorrect. Actually Liénard considered only the case $g(x)=x$. The proof given by the author is due to Levinson and Smith [Duke Math. J. 9, 382–403 (1942); these Rev. 4, 42], who did, however, call the more general equation the Liénard equation.] *N. Levinson.*

Cartwright, M. L., and Littlewood, J. E. On non-linear differential equations of the second order. I. The equation $\ddot{y}-k(1-y^2)\dot{y}+y=b\lambda k\cos(\lambda t+a)$, k large. J. London Math. Soc. 20, 180–189 (1945).

The equation appearing in the title is considered in this preliminary survey. For $b>\frac{2}{3}$ and k large the equation has a periodic solution of period $2\pi/\lambda$ toward which all other solutions tend. If $b<\frac{2}{3}$ great complication in the stationary solutions can and does arise. In particular, discontinuous recurrent motions occur.

The transformation T of the $(y, \dot{y})$ plane into itself is defined as taking initial points of solutions of the equation into the final point over an interval in t from t_0 to $t_0+2\pi/\lambda$. Under T, for some values of $0<b<\frac{2}{3}$, the plane is divided into two parts, an interior and exterior, by a connected set K_0 of zero area which is invariant under T. The complication in the structure of K_0 is very great. While G. D. Birkhoff has shown the existence of analytic transformations of the plane into itself with a complicated invariant curve K_0, this is the first time that it has been shown that the situation can arise from the transformation associated with a simple differential equation. Hopes of the reviewer that singular invariant curves such as K_0 could be ruled out by some reasonable regularity requirement [Ann. of Math. (2) 45, 723–737 (1944); these Rev. 6, 173] are completely destroyed. It is shown that there are stable periodic sub-

harmonic motions and unstable ones as well as discontinuous recurrent motions each associated with some subset of K_b.

For other values of b, $0<b<\frac{2}{3}$, a much simpler situation occurs. The values of b, $0<b<\frac{2}{3}$, about which nothing can be said concerning the stationary solution can be made a set of arbitrarily small measure by taking k large enough.
N. Levinson (Cambridge, Mass.).

Kazakevich, V. V. On approximate integration of van der Pohl's equation. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 414–417 (1945).

Kazakevitch, V. V. Sur l'intégration approximative des systèmes oscillatoires à un degré de liberté. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 107–110 (1946).

Kasakevitch, V. V. Sur le processus d'établissement de systèmes d'oscillation à un degré de liberté. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 486–489 (1945).

In the first paper the author derives an approximate formula for the period of oscillation of the periodic solution of van der Pol's equation $\ddot{x}+\mu(x^2-1)\dot{x}+x=0$ for arbitrary positive values of μ. No estimates are given as to the order of accuracy of these approximate values, but a chart of the values $\mu=1, 5, 10$ shows an agreement to within 10% with the values obtained by means of graphical analysis by van der Pol. In the other papers the method is extended to treat the general equation $\ddot{x}+f(x, \dot{x})=0$. *R. Bellman.*

Volk, I. M. Elastic oscillations with damping proportional to a power of velocity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 125–134 (1946). (Russian. English summary) [MF 16839]

The paper deals with the motion of a system with one degree of freedom, described by a differential equation $\ddot{x}+q^2x+p\dot{x}^n=0$, $0\leqq t\leqq T$, with initial conditions $x(0)=-a$, $\dot{x}(0)=0$. *S. Lefschetz* (Princeton, N. J.).

Volk, I. M. A generalization of the method of small parameter in the theory of non-linear oscillations of non-autonomous systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 437–440 (1946).

The author applies the small parameter method of Poincaré to the system $dx_s/dt=X_s(x_1, \cdots, x_n, \mu, t)$, $s=1, \cdots, n$, where the X_s are periodic functions of t, analytic functions of the x_k and meromorphic functions of μ within a domain $a\leqq|x_k|\leqq b$, $|\mu|\leqq\rho$. Results are obtained analogous to those obtained by Poincaré for the case where the X_s are analytic functions of μ. *R. Bellman* (Princeton, N. J.).

Teodorčik, K. The laws of co-existence of frequencies in soft auto-oscillating systems. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1946, 385–388 (1946). (Russian)

In mechanical systems with n degrees of freedom the condition of self-excitation can be fulfilled simultaneously by not more than n distinct oscillations. The question arises: for which m of these ($m\leqq n$) do the self-excited oscillations become stationary? The author obtains results which answer this question in certain cases. *S. Lefschetz.*

Mangeron, D. I. Mécanique non-linéaire sur les systèmes oscillatoires non linéaires. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 62–66 (1946).

The author treats a second order nonlinear equation in which the independent variable does not appear explicitly by leaving the linear terms with constant coefficient on the left and putting the nonlinear terms on the right. The application of the usual variation of constants formula recasts the problem as a nonlinear Volterra integral equation to which successive approximations can be applied. [This method has been used before.] *N. Levinson.*

Rocard, Yves. Les méfaits du roulement. Autooscillations et instabilités de route. Revue Sci. 84, 15–28 (1946).

This is a discussion of various effects, such as resonances and self-excited oscillations, which may disturb the motions of automobiles and other vehicles. The basic physical discussion is rather thorough. The mathematical theory is based upon the simplifying assumption that the variables representing the disturbance of the motion are small, so that the differential equations of motion are linear with constant coefficients.

The emphasis is upon the possibility of instability and self-excited oscillations. This possibility arises from the gyroscopic forces exerted by the rotating wheels and from the properties of the couplings between various degrees of freedom which are established by the rolling of the elastic tires on the pavement. The author discusses conditions upon the speed and mechanical structure of the vehicle which insure stability of the motion. One important self-excited oscillation, called "shimmy," consists of an oscillation of the front wheels about vertical axes. The author criticizes Den Hartog's attempted explanation of this phenomenon [Mechanical Vibrations, 2d ed., McGraw-Hill, New York, 1940, pp. 372–375] and offers an alternative one of his own.
L. A. MacColl (New York, N. Y.).

Abelé, Jean. Les oscillations de relaxation et le problème de leur définition analytique. Rev. Gén. Sci. Pures Appl. 53, 68–77 (1946).

Rosenblatt, Alfred. On subharmonic resonance. Actas Acad. Ci. Lima 8, 45–58 (1945). (Spanish)

Consider a nonlinear dynamical system subjected to a force which is a sinusoidal function of time with the angular frequency ω. Under certain circumstances the response of the system may be periodic with a fundamental frequency which is smaller than ω. This phenomenon is called subharmonic resonance. The present note gives an exposition of the usual perturbation theory of the phenomenon, with a rather detailed discussion of one simple example. There appear to be no essentially new results. *L. A. MacColl.*

Haag, Jules. Sur la synchronisation sous-harmonique. C. R. Acad. Sci. Paris 223, 525–527 (1946).

A brief statement, without proofs, of some results concerning the response of a nonlinear dynamical system to a periodic impressed force. Most of the results relate to the existence and stability of subharmonic oscillations. The methods appear to be appropriate variations of methods which are already well known. To what extent the results are essentially new is not clear. The author emphasizes the mathematical rigor of his discussion. *L. A. MacColl.*

Haag, Jules. Sur certaines systèmes différentiels à solutions périodiques. C. R. Acad. Sci. Paris 223, 446–449 (1946).

The author considers a linear system, with periodic coefficients of period T, (1) $\dot{x}_i=\lambda\sum p_{ij}(t)x_j$, $i=1, \cdots, m$; λ is a constant. Let $a_{ij}=\int_0^T p_{ij}(t)dt$. Let r_k, $k=1, \cdots, m$, be the

roots of the characteristic determinant associated with the a_{ij}. Then, as $\lambda \to 0$, the author states that the characteristic exponents of (1) are asymptotic to λr_k. Theorems on stability are also stated for the nonhomogeneous case of (1) with $\lambda f_i(t) + \lambda g_i(x_k, t)$ added to the right member. These results require the real parts of r_k to be negative and, among other things, that f and g tend to zero or to periodic functions as $t \to +\infty$. *N. Levinson* (Cambridge, Mass.).

Bellman, Richard. On the stability of systems of differential equations. Proc. Nat. Acad. Sci. U. S. A. **32**, 190–193 (1946). [MF 16862]

Three theorems relating the behavior of a perturbed system of differential equations to an unperturbed linear system are stated. As stated, theorems I and II are incorrect (but can be corrected). Therefore the statement of the results will be left for the review of the longer account which the author will publish later. *N. Levinson.*

Borg, Göran. Über die Stabilität gewisser Klassen von linearen Differentialgleichungen. Ark. Mat. Astr. Fys. **31A**, no. 1, 31 pp. (1944).

The equation $y'' + \{\alpha + \beta\psi(x)\}y = 0$, where $\psi(x)$ is periodic of period π, is said to have stable solutions if the characteristic exponents, determined from Flouquet's theory, are purely imaginary. The conditions

$$\int_0^\pi \psi(x)dx = 0, \qquad \left\{\pi^{-1}\int_0^\pi |\psi(x)|^p dx\right\}^{1/p} = 1, \qquad p \geqq 1,$$

are considered for three cases, $p=1$, 2 and ∞. In each case the region in the (α, β)-plane is determined where all solutions of the differential equation are stable. The author's result for $p=1$ is as follows. The region of stability in the (α, β)-plane for the case $p=1$ is bounded by the curves $\beta_{n+1} = \pm 4\pi^{-1}(n+1)\alpha^{\frac{1}{2}} \cot \{\frac{1}{2}\alpha^{\frac{1}{2}}\pi/(n+1)\}$, $n^2 \leqq \alpha \leqq (n+1)^2$, $n = 0, 1, 2, \cdots$; $\bar{\beta}_n = \pm 2\alpha(1 - n\alpha^{-\frac{1}{2}})$, $\alpha > 1$, $n \geqq 1$; $\alpha = 0$; and contains none of these curves in its interior. Similar results are given for the cases $p=2$ and $p=\infty$. *N. Levinson.*

Četajev, N. G. Calculation of particular solutions for systems of linear differential equations with constant coefficients. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 291–294 (1946). (Russian. English summary) [MF 16850]

The author adapts D'Alembert's method of integrating linear equations with constant coefficients to systems

$$dx_i/dt = p_{i1}x_1 + \cdots + p_{in}x_n, \qquad i = 1, \cdots, n,$$

where the p_{ij} are real constants. A method is given for the determination of particular solutions. With the aid of Liapounoff's developments one may find the normal or canonical variables for the system. *W. J. Trjitzinsky.*

Bulgakov, B. V. On the accumulation of disturbances in linear oscillatory systems with constant parameters. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 343–345 (1946).

The author considers the system

$$\sum_{i=1}^{N} f_{ij}(D)x = \sum_{\nu=1}^{n} g_{j\nu}(D)y_\nu(t), \qquad j = 1, \cdots, N,$$

where $D = d/dt$, and obtains bounds on the solutions in terms of bounds on the $y_\nu(t)$ under various assumptions concerning the matrix $(f_{ij}(D))$. *R. Bellman.*

Bulgakov, B. V. On normal coordinates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 273–290 (1946). (Russian. English summary) [MF 16849]

The author examines systems of the form $f(D)y = \varphi(y, t)$, where y is an n-dimensional column vector and $f(D)$ is a matrix whose terms are polynomials in $D = d/dt$. His purpose is to reduce the system to the normal form $Dy = \psi(y, t)$ without introducing new variables. This is done directly by assuming that (1) the determinant of the operational matrix $f(D)$ is not identically zero; (2) all the elementary divisors of $f(D)$ are linear. *S. Lefschetz* (Princeton, N. J.).

Solodovnikov, V. V. On an approximate method of investigation of the dynamics of a regulating system or a following system. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 1179–1202 (1945). (Russian)

In the author's own words, "this paper makes no pretense at mathematical rigor but merely indicates a possible way of developing an approximate method for a qualitative study of differential equations." The author considers a transient process governed by an equation of the form $a_n\delta^{(n)}(t) + \cdots + a_1\delta'(t) + a_0\delta(t) = f(t)$, a_i = constant, whose solutions may be represented in the form

$$\delta(t) = (2/\pi)\int_0^\infty \omega^{-1}P(\omega) \sin \omega t dt,$$

where the "frequency characteristic"

$$P(\omega) = \Re\left\{i\omega \int_0^\infty e^{-i\omega t}\delta(t)dt\right\}$$

can be computed from the differential equation and the initial conditions. Using known theorems as well as plausibility considerations he formulates rules of thumb which permit him to predict whether or not the function $\delta(t)$ corresponding to a given function $P(\omega)$ satisfies the following conditions: (I) $\lim_{t\to+\infty} \delta(t) = \delta_0$ and $\int_0^\infty |\delta(t) - \delta_0| dt < \infty$, (II) $|\delta(t)| \leqq \delta_m$ for $t > 0$, (III) for $t \geqq t_0$, $|\delta(t) - \delta_0| \leqq \Delta$, (IV) $\int_0^{t_0}(1 + \delta'(t))^{\frac{1}{2}}dt \leqq l_0$, where δ_0, δ_m, Δ, t_0 and l_0 are given numbers. The last condition serves to limit the number of changes in sign of $\delta(t) - \delta_0$.

The method is illustrated by several numerical examples. It is shown how similar considerations may be used in order to replace a complicated differential equation by a simpler one without causing a significant change in the solution. *L. Bers* (Syracuse, N. Y.).

Firsov, G. A. On the question of the stiffness of a ship under the influence of a squall. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 648–656 (1945). (Russian)

An endeavor to investigate in detail the motion of a ship in a squall, taking into account the time interval during which the load due to wind is increasing as well as the increase in heeling moment due to the increase in drift. *S. Lefschetz* (Princeton, N. J.).

Firsov, G. A. On the question of the oscillation of a ship provided with an active stabilizer. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 995–1002 (1945). (Russian)

The author presents theoretical investigations of the oscillations of a ship with an active regulator whose action is expressed by an odd periodic function. The assumptions are the usual ones applied in questions of rolling and due

mainly to Froude. They are, in particular, (a) resistance to oscillations is proportional to the first power of angular velocity, (b) the wave is assumed straight and strictly harmonic, (c) the transverse proportions of the ship are small relative to the wave and pressure in the wave is hydrostatic, (d) the angles of deviation from the vertical are small, (e) only the steady state is considered. The investigation leads to practical conclusions about the effect on the efficiency of the regulator of the water resistance, the static characteristic of the regulator and the rate of growth of restoring moment in relation to the counter-heeling moment. *S. Lefschetz* (Princeton, N. J.).

Haskind, M. D. The oscillation of a ship in still water. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 23–34 (1946). (Russian)

The author investigates the general motion of a floating body in still water. He starts with the fundamental equations of hydrodynamics but assumes linearization. The relations obtained are applied to a ship with particular attention to pitching and up-and-down motion. It is found in certain cases that pitching and vertical oscillations are not independent. *S. Lefschetz* (Princeton, N. J.).

Wintner, Aurel. The adiabatic linear oscillator. Amer. J. Math. **68**, 385–397 (1946).

The author considers the equation $x''+(1+\phi(t))x=0$ subject to (a) $\phi(t)\to 0$ as $t\to+\infty$ or (b) $\int^\infty|\phi(t)|dt<\infty$. Using Sturmian considerations, it is possible under (a) to speak of the nth amplitude a_n of a solution $x(t)$. The author proves that $a_{n+1}/a_n\to 1$ as $n\to\infty$, and similar results. A systematic procedure is given for setting up examples having interesting behavior with (a) satisfied. *N. Levinson.*

Wintner, Aurel. An Abelian lemma concerning asymptotic equilibria. Amer. J. Math. **68**, 451–454 (1946).

Let $f=(f_1, \cdots, f_n)$ and $x=(x_1, \cdots, x_n)$ denote n-dimensional vectors. Let $f=f(t, x)$ be a continuous function on $0\leqq t<\infty$, $-\infty<x_i<\infty$, $i=1, \cdots, n$, for which there exists a pair of functions (of a single real variable) $\lambda=\lambda(r)$ and $\phi=\phi(r)$, defined and continuous for $0\leqq r<\infty$, positive for $0<r<\infty$, satisfying $|f(t, x)|<\lambda(t)\phi(|x|)$, $\int^\infty\lambda(t)dt<\infty$ and $\int^\infty\phi(t)dt=\infty$. Then, if $x=x(t)$, defined for some interval $0\leqq t<\epsilon$, is a solution of the differential equation $x'=f(t, x)$, the definition of $x(t)$ can be extended over the entire half-axis $0\leqq t<\infty$ so that $x(t)$ remains a solution; for any such extension (which need not be unique), $\lim_{t\to\infty} x(t)$ exists. This improvement of an earlier result of the author [same J. 68, 125–132 (1946); these Rev. 7, 297] is obtained with a simple proof which does not involve the method of successive approximations previously employed. *P. Hartman.*

Wintner, Aurel. Linear variations of constants. Amer. J. Math. **68**, 185–213 (1946). [MF 16416]

The notion of a cross modulus of a matrix is introduced as follows. Given a matrix $A=(a_{ik})$, let $|A|$ be the greatest of the $|a_{ik}|$; if $[X]$ is the matrix $RX-XR$, one has (1) $|[X]|\leqq r|X|$, where the number r is independent of X; the least r satisfying (1) is the cross modulus of R. Some of the results are as follows. Let the cross modulus of a constant matrix R be less than 1 and let $G(t)$ be a matrix function, continuous and bounded for $0<t\leqq t_0$; then a solution matrix of $x'=F(t)x$ $(F(t)=R/t+G(t))$ is a matrix product of the form $P(t)t^R$, where $P(+0)$ exists, $P(+0)\neq\infty$, $\det P(+0)\neq 0$. Rules are given for the variation of constants from either side. If $A(t)$ is continuous for $0<t\leqq t_0$ and is absolutely integrable and if $X(t)$ is a fundamental matrix of $x'=A(t)x$, then $X(+0)$ exists and is finite. Suppose that every solution x of (2) $x'=A(t)x$, where A is continuous on a half line, is bounded, as $t\to\infty$, and $\liminf \int_{t_0}^t \operatorname{tr} A(s)ds>-\infty$; let B be continuous and $\int^\infty|B-A|dt<\infty$; then for every solution x of (2) there exists a solution y of $y'=B(t)y$ such that $y-x\to 0$ as $t\to\infty$; the correspondence between the solutions x, y is continuous in terms of the initial data $x(t_0)$, $y(t_0)$. The importance of the paper lies in the fact that the first of the above results can be generalized in an abstract direction. *W. J. Trjitzinsky* (Urbana, Ill.).

Hartman, Philip. On the solutions of an ordinary differential equation near a singular point. Amer. J. Math. **68**, 495–504 (1946).

This is a continuation of the considerations of a recent paper by Hartman and Wintner [same J. 68, 301–308 (1946); these Rev. 7, 444] dealing with the solutions of an equation of the form (1) $\varphi(x)y'=f(x, y)$. The emphasis is on the behavior of the ratio $y(x)/x$ as $x\to 0$. Four theorems, with rather complicated statements, are proved. The first of these can be stated as follows. Let $\varphi(x)$ be a positive continuous function on the interval $0<x\leqq a$, satisfying the condition $\varphi(x)/x\to 0$ as $x\to+0$. Let $f(x, y)$ be a real-valued continuous function on the partly open rectangle $R: 0<x\leqq a$, $|y|\leqq b$, and satisfy there the lower Lipschitz condition $|\{f(x, y_1)-f(x, y_2)\}/(y_1-y_2)|>c>0$. Let there exist a function $y=y^0(x)$, $0<x\leqq a$, satisfying the relations $|y^0(x)|<b$, $f(x, y^0(x))\equiv 0$, and $\lim_{x\to+0} y^0(x)/x=\gamma$ exists. If $y=y(x)$ is any solution path of (1) satisfying the condition $\lim_{x\to+0} y(x)=0$, then $\lim_{x\to 0} y(x)/x=\gamma$. *L. A. MacColl.*

Rosenblatt, Alfred. On the developments in series of the solutions of differential equations of the first order and first degree in a neighborhood of an essential singular point. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 1, 111–127 (1946). (Spanish)

This paper is concerned with solutions of equations of the form $x^n dy/dx=ay^m+yf(x, y)$, $a\neq 0$, where $f(x, y)$ is a function of class C^2 such that

$$f(x, y)=xy\psi(x, y)+cy^p\{1+f(y)\}+bx^h\{1+\varphi(x)\},$$

and m, n, h and p are integers satisfying the conditions $n>1$, $m>1$, $p\geqq m$, $h\geqq 1$, $h\geqq n-1$ and $n-1>(n-2)(m-1)$. The title of the paper is somewhat misleading, for no developments of the solutions in series, in the ordinary sense, are given. Instead, the author uses Picard's method of successive approximations to establish the existence of solutions in a variety of subcases and to study some of the properties of those solutions. In the cases in which $m\geqq 2$ and $h>n-1$ all of the solutions which pass through points (x_0, y_0) sufficiently near the origin pass through the origin. In the case in which $h=n-1$ and $(1-m)b+n-1>0$ a similar result is obtained, subject to some further restrictions on the position of the initial point (x_0, y_0).

L. A. MacColl (New York, N. Y.).

Blanc, Ch. Les séries de Fourier et leur application à certaines intégrations. Bull. Tech. Suisse Romande **69**, 19 pp. (1943).

The Fourier coefficients of a function with jumps are expressed in terms of the Fourier coefficients of its integral and the jumps. The resulting formulas are shown to lead to simple solutions of some boundary value problems for ordinary differential equations. *R. P. Boas, Jr.*

Blanc, Charles. Sur les équations différentielles linéaires non homogènes, à coefficients constants. Comment. Math. Helv. 19, 61–71 (1946).

The equation $DU=U^{(n)}+a_1U^{(n-1)}+\cdots+a_nU=F(t)$ is studied, where $a_1, \cdots, a_n$ are constants and where all solutions of the homogeneous equation $DU=0$ approach zero as $t\to+\infty$. Under the assumption that $F(t)$ is analytic for real values of t, a solution of the nonhomogeneous equation has been expressed in terms of $F(t)$ and its derivatives [J. R. Carson and T. C. Fry, Bell System Tech. J. 16, 513–540 (1937)]. The author obtains expressions for the solutions of the equation $DU=F(t)$ in cases where the function $F(t)$ is not assumed analytic. These are analogous to Taylor's formula with remainder in the Lagrange form. The most general expressions apply to cases where $F(t)$ is assumed integrable and bounded and they are then shown to become more explicit as further assumptions are made on $F(t)$. In particular, it is shown that, when $F(t)=e^{\lambda t}K(t)$, where $K(t)$ is analytic for real values of t with the exception of a sequence of values $t_0, \cdots, t_k$, $t_k\to\infty$, and $K^{(r)}(t)$ is always interior to a rectangle which is itself interior to a circle $|z|=A^r$, $0<A<\Re\lambda+\rho$, $-\rho$ being the greatest of the real parts of the roots of the characteristic equation (all of which are assumed negative), then the solution $U(t)$ is the sum of a convergent infinite series. The explicit form of this series is given. *W. M. Whyburn* (Lubbock, Tex.).

Aprile, Giuseppe. Sull'introduzione delle "condizioni iniziali" nel calcolo operazionale dei sistemi fisici retti da leggi lineari. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 243–246 (1942).

Riordan, Nelson F., and Waidelich, D. L. Discussion on "The steady-state operational calculus." Proc. I.R.E. and Waves and Electrons 34, 579–580 (1946).

The paper by Waidelich quoted in the title appeared in the same vol., 78P–83P (1946); these Rev. 7, 297.

★Herreng, Pierre. Les applications du calcul opérationnel. École Norm. Sup. Publ. Laboratoires. Physique, no. 6. Masson et Cie., Paris, 1944. 91 pp.

A detailed study of the applications of the Laplace transform to ordinary and partial differential equations, particularly those of electricity. *H. Pollard* (Ithaca, N. Y.).

Kimball, W. S. The Foucault pendulum star path and the n-leaved rose. Amer. J. Phys. 13, 271–277 (1945).

This paper contains a detailed and elementary discussion of certain solutions of the linear system $\ddot{x}-2\omega_L\dot{y}=-\omega_P{}^2x$, $\ddot{y}+2\omega_L\dot{x}=-\omega_P{}^2y$, where ω_L and ω_P are certain constants. These equations, in the author's application to the Foucault pendulum and with his notation, should be corrected by subtracting $\omega_L{}^2x$ and $\omega_L{}^2y$, respectively, from the left hand members. Since ω_L is quite small compared with ω_P, this does not greatly affect the validity of the results.
D. C. Lewis (College Park, Md.).

Popoff, Kyrille. Sugli integrali di alcune equazioni differenziali considerate come funzioni dei parametri che vi figurano, per grandi valori dei parametri. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 524–531 (1942).

The equation $y''+kP(x)y'+kQ(x)y+kS(x)=0$ is typical of the equations which the author has under consideration. The paper is concerned with a method for studying the solutions when the parameter k is large. The solution considered is assumed to satisfy initial conditions of the form $y(0)=a$, $y'(0)=b$. The equation is first written in the form

$$y'=-Ry-(u/k)y''-T, \tag{1}$$

where $R=Q/P$, $T=S/P$, $u=1/P$. The expression $uy''/k+T$ is treated momentarily as a known function and (1) is solved for y as a linear differential equation of the first order. Again, the original equation is written in the form

$$dy'/dx=-kPy'-kQy-kS, \tag{2}$$

$kQy+kS$ is treated as a known function, and (2) is solved for y' as a linear equation of the first order. Then elementary manipulation leads to an integral equation of the form

$$y(x)=F(x)-\int_0^x N(x,s)y(s)ds,$$

where $F(x)$ and $N(x,s)$ are complicated functions which, however, involve k in such ways that the determination of the behavior of $y(x)$ for large values of k is easy.

This basic procedure applies when $P(x)$ is positive throughout the range of values of x under consideration. Modifications are necessary when $P(x)$ is not always of one sign. *L. A. MacColl* (New York, N. Y.).

Cinquini, Silvio. Sopra un'osservazione del Signor Scorza-Dragoni su un problema per le equazioni differenziali ordinarie. Ann. Scuola Norm. Super. Pisa (2) 11, 217–221 (1942). [MF 16762]

Caligo, Domenico. Complementi alla valutazione asintotica delle funzioni di Sturm-Liouville. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 643–650 (1942).

The author discusses the first term in the asymptotic expansion of the eigenvalues and eigenfunctions of the equation $u''+\{\lambda^2-q(x)\}u=0$, subject to the usual boundary conditions. He supposes that $q(x)$ is continuous or satisfies a Hölder condition. He is evidently unaware of more precise results already in the literature [cf., for example, E. L. Ince, Ordinary Differential Equations, Longmans, Green, London, 1927, pp. 271–273]. *R. P. Boas, Jr.*

Bobonis, Augusto. A sufficiency theorem for differential systems. Bull. Amer. Math. Soc. 52, 465–474 (1946). [MF 16803]

This paper treats a differential system

$$(*)\qquad \begin{aligned} &y'(x)=[A(x)+\lambda B(x)]y(x), \qquad a\leqq x\leqq b,\\ &(M^0+\lambda M^1)y(a)+(N^0+\lambda N^1)y(b)=0,\end{aligned}$$

where $y(x)$ is a vector and $A(x)$, $B(x)$, M^0, M^1, N^0, N^1 are square matrices, which is definitely self-adjoint in the sense defined by the author in his dissertation [Contributions to the Calculus of Variations, 1938–41, University of Chicago Press, 1942, pp. 99–138; these Rev. 4, 200] and which satisfies the additional condition that $B(x)$ is of constant rank on $a\leqq x\leqq b$. The author extends the method used by Reid [Trans. Amer. Math. Soc. 44, 508–521 (1938)], and from the extremal properties of the characteristic values of an equivalent boundary problem deduces a necessary and sufficient condition for the system (*) to have infinitely many characteristic values. *W. T. Reid* (Evanston, Ill.).

Stokalo, I. Generalization of Heaviside's formula to a case of linear differential equations with variable coefficients. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 339–340 (1946).

This note is related to one by the author in the same C. R. 47, 10–11 (1945) [these Rev. 7, 298]. It announces results in the following terms, the proofs being omitted. (I) For

the system of differential equations

$$dx(t)/dt = A(t)x(t) + \int_{a-i\infty}^{a+i\infty} e^{pt}\varphi(p)dp,$$

in which $\varphi(p)$ is a matrix that is regular for $|\Re(p)| > g$ and such that $\int_{-\infty}^{\infty} |\varphi(a+iy)| dy$ exists and is uniformly bounded for $|a| \geqq g$, the solution satisfying the condition $x(0) = E$ (the unit matrix) can be represented in the form

$$x(t) = (1/2\pi i)\int_{a-i\infty}^{a+i\infty} e^{pt}\omega(t, p)\varphi(p)dp.$$

(II) The expression

$$x(t) = (1/2\pi i)\int_{a-i\infty}^{a+i\infty} e^{pt}\omega(t, p)dp$$

gives the solution of $dx(t)/dt = A(t)x(t)$ for which $x(t) \equiv 0$ $(t<\tau)$, $x(\tau+0) = E$. (III) For the system

$$dx(t)/dt = A(t)x(t) + F(t),$$

in which $F(t)$ is a matrix, satisfying Dirichlet's condition on the whole real axis, such that the integral $\int_0^\infty e^{-p\tau}F(\tau)d\tau$ converges absolutely, the solution for which

$$x(t) \equiv 0 \ (t<\tau), \quad x(\tau+0) = E$$

is

$$x(t) = (1/2\pi i)\int_{a-i\infty}^{a+i\infty} e^{p(t-\tau)}\omega(t, p)\psi(p)dp,$$

where

$$\psi(p) = \int_0^\infty e^{-p\tau}F(\tau)d\tau.$$

R. E. Langer (Madison, Wis.).

Petrov, B. N. The limits of applicability of S. Tchaplygin's theorem on differential inequalities to linear equations with usual derivatives of the second order. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 255–258 (1946).

If $y(x)$ is a solution of $L(y) \equiv y'' - p_1y' - p_2y - q = 0$ such that $y(x_0) = y_0$, $y'(x_0) = y_0'$ and if $v(x)$ is such that $L(v) > 0$, $v(x_0) = y_0$, $v'(x_0) = y_0'$, then one can conclude that $v(x) > y(x)$ if $x_0 < x \leqq x_1$ provided that there is a continuous solution $\lambda(x)$ of the Riccati equation $\lambda' + \lambda^2 + p_1\lambda + (p_1' - p_2) = 0$ on the same interval. The largest such number x_1 is determined in case p_1 and p_2 are constants and is ∞ if $p_1^2 + 4p_2 \geqq 0$, but is finite otherwise. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Petrov, B. N. Inapplicability of the theorem on the differential inequality of S. Tchaplygin to certain non-linear differential equations of the second order. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 497–499 (1946).

Let $y(x)$ be a solution of the equation $y'' = f(x, y, y')$, such that $y(x_0) = y_0$, $y'(x_0) = y_0'$. The author shows that when $f = -y^2$ there exists a function $v(x)$ such that $v'' > f(x, v, v')$, $v(x_0) = y_0$, $v'(x_0) = y_0'$, for which $v(x) - y(x)$ changes sign at a point arbitrarily close to x_0. This cannot happen if $f(x, y, y')$ is linear in y and y'. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Lusin, N., et Kouznetzoff, P. Sur l'invariabilité absolue et l'invariabilité à ϵ près dans la théorie des équations différentielles. I. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 251–253 (1946).

Lusin, N., et Kouznetzoff, P. Sur l'invariabilité absolue et l'invariabilité à ϵ près dans la théorie des équations différentielles. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 335–337 (1946).

Consider the system of differential equations

$$(1) \qquad \sum_{j=1}^{n} a_{ij}x_j = \delta_i f(t). \qquad i = 1, \cdots, n,$$

where a_{ij} is a polynomial with constant coefficients and of second degree in D, $D = d/dt$, $\delta_1 = 1$, $\delta_2 = \cdots = \delta_n = 0$. The function $f(t)$ is assumed analytic and the determinant $\Delta(D) = |a_{ij}|$ is not identically zero. In paper I it is proved that the vanishing of the minor of a_{11} in Δ is a necessary and sufficient condition for $x_1(t)$ to be independent of the function $f(t)$. If the first $2n-2$ derivatives of $f(t)$ are bounded on $-\infty < t < \infty$ and if the real parts of the roots of $\Delta(\alpha) = 0$ are all negative, paper II shows that the system (1) has only one bounded solution on this interval. If $E_1(t), \cdots, E_n(t)$ represents this bounded solution, and $x_1(t), \cdots, x_n(t)$ represents any other solution of (1), then for every $\epsilon > 0$ there exists a T such that $|E_i(t) - x_i(t)| < \epsilon$, $t > T$, $i = 1, \cdots, n$.
F. G. Dressel (Durham, N. C.).

Sansone, Giovanni. Le equazioni differenziali lineari, omogenee, del quarto ordine, nel campo reale. Ann. Scuola Norm. Super. Pisa (2) **11**, 151–195 (1942). [MF 16759]

Sansone, G. Studio degli integrali del sistema $y'' + py = qz$, $z'' + pz = ry + \omega y'$. Ann. Mat. Pura Appl. (4) **22**, 145–180 (1943).

In the first paper the author shows that under suitable differentiability conditions on the coefficients the linear differential equation of the fourth order

$$y^{(iv)} + 4p_1(x)y''' + 6p_2(x)y'' + 4p_3(x)y' + p_4(x)y = 0$$

is reducible to the form

$$(\alpha) \qquad [\theta_2(x)y'']'' - [\theta_1(x)y']' - \omega(x)y' + \theta_0(x)y = 0,$$

and that, in turn, (α) is equivalent to a second order system of the form

$$(\beta) \qquad y'' + p(x)y = q(x)z, \quad z'' + p(x)z = r(x)y + \omega(x)z.$$

For (α) there are established some separation and comparison theorems, together with results on the asymptotic behavior of solutions on an infinite interval. Analogous problems for (β) are treated in the second paper. None of the results will be stated specifically here, in view of the rather complicated hypotheses involved. In particular, however, various of the results extend to the general equation (α) or system (β) results previously known for the self-adjoint case in which $\omega(x) \equiv 0$ [see, in particular, G. Cimmino, Math. Z. **32**, 4–58 (1930); W. M. Whyburn, Amer. J. Math. **52**, 171–196 (1930)]. The author also treats briefly equations (α) and systems (β) with coefficients depending upon a parameter; the results for these cases are primarily concerned with criteria for oscillatory solutions. *W. T. Reid.*

Chiellini, Armando. Sugli invarianti del sistema differenziale formato da due equazioni lineari omogenee del secondo ordine. Rend. Sem. Fac. Sci. Univ. Cagliari **10**, 109–120 (1940). [MF 16216]

Selon Wilczynski le système plus général de deux équations différentielles linéaires homogènes du deuxième ordre en deux fonctions inconnues se transforme, par une substitution linéaire sur les fonctions, dans la forme normale

$$y'' + p_{12}z' + q_{11}y + \tfrac{1}{2}p_{12}'z = 0, \quad z'' + p_{21}y' + q_{22}z + \tfrac{1}{2}p_{21}'y = 0.$$

L'auteur appelle cette forme "réduite alterne." En partant d'elle, il cherche un système d'invariants qui tienne compte aussi d'un changement de variable; il trouve que les fonctions p_{12}, p_{21}, $q_{11} - q_{22}$, $4p_{12}^2q_{11} + 3(p_{12}')^2 - 2p_{12}p_{12}''$ forment un système complet des invariants relatifs se modifiant seulement par un facteur, puissance de la dérivée de la fonction qui définit le changement de variable. La recherche termine avec l'application au cas des systèmes qui admettent une

réduite alterne à coefficients constants et à quelques autres exemples. Elle se trouve un peu compliquée par des calculs qui nous semblent superflus. Diverses fautes d'impression affectent les formules.
B. Levi (Rosario).

Graff, A. A. To the theory of linear differential systems in one-dimensional domain. Rec. Math. [Mat. Sbornik] N.S. 18(60), 305–328 (1946). (Russian. English summary)

A study is given of linear differential systems, constructed for quasi-differential operators of the form

$$D_n y = \rho_n \frac{d}{dx} \cdots \frac{d}{dx} \rho_0 y.$$

The definition and some properties of adjoint systems of integrals are presented. The boundary conditions involved are "polylocal" or of "integral" type. Some properties of the self-adjoint system of integrals of the self-adjoint equation

$$D_{2n} y \equiv \rho_0 \frac{d}{dx} \cdots \frac{d}{dx} \rho_n \frac{d}{dx} \cdots \frac{d}{dx} \rho_0 y = 0$$

are also established. The method involves a generalization of Kneser's method of constructing the Green's function for the Sturm-Liouville differential operator, under Sturmian boundary conditions. The key to the success of the method is in the introduction of an appropriate concept of an adjoint problem, a concept distinct from that of Bunitzky, Tamarkin, etc. The results of the work can be extended to differential systems belonging to an operator of the type

$$L_n(y) = \rho_{nn}(d/dx)L_{n-1}(y) + \sum_{j=1}^{n-1} \rho_{nj} L_j(y),$$

where $L_0(y) = \rho_{00} y$, $L_k(y) = \rho_{kk}(d/dx)L_{k-1}(y) + \sum_1^{k-1} \rho_{kj} L_j(y)$.
W. J. Trjitzinsky (Urbana, Ill.).

Germay, R. H. J. Extension de la théorie des intégrales premières aux systèmes complètement intégrables d'équations aux différentielles totales. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 86–92 (1946).

Let

$$(*) \qquad dz_i = \sum_{j=1}^{n} a_{ij}(x_1, \cdots, x_n; z_1, \cdots, z_p)dx_j, \quad i = 1, \cdots, p,$$

be a completely integrable pth order system of total differential equations. Then $F(x_1, \cdots, x_n; z_1, \cdots, z_p)$ is called a first integral of (*) if (**) $F(x_1, \cdots, x_n; z_1, \cdots, z_p) = c$ when the z_i are replaced by the solutions of (*). The paper shows that, if a first integral (**) is known, the pth order system (*) can be reduced to a complete system of order $p-1$. Furthermore, the solution of this system of order $p-1$ together with (**) furnishes the solution of (*).
F. G. Dressel (Durham, N. C.).

van der Kulk, W. Contributions to the theory of the $\mathfrak{S}_d^m$-field. IV. Conditions for complete integrability. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 575–583 (1943). (Dutch. German, English and French summaries) [MF 15782]

[The first three parts appeared in Nederl. Akad. Wetensch., Proc. 44, 452–463, 625–635 (1941); 45, 26–31 (1942); these Rev. 3, 43; 6, 66.] An $\mathfrak{S}_d^m$ is a d-dimensional manifold of m-dimensional linear elements E_m at a point of a general manifold X_n. A field of such $\mathfrak{S}_d^m$, defined at different points of an X_n, is studied with the aid of parametric equations; a correspondence is set up between the $\mathfrak{S}_d^m$-field and an E_{m+d}-field in a certain X_{n+d}. Complete integrability of an $\mathfrak{S}_d^m$-field corresponds to the property that there exists through each point of the X_{n+d} at least one integral X_m of a particular kind of the E_{m+d}-field. With the aid of this property it is possible to prove three theorems which were announced in the preceding parts and deal with the complete integrability of $\mathfrak{S}_d^m$-fields. For this purpose the theory of Cartan is applied to E_{m+d}-fields [É. Cartan, Ann. Sci. École Norm. Sup. (3) 18, 241–311 (1901)]. The proof of the theorem announced in part III is fully presented here.
D. J. Struik (Cambridge, Mass.).

van der Kulk, W. Contributions to the theory of the $\mathfrak{S}_d^m$-fields. V. Contact transformations of $\mathfrak{S}_d^m$-fields. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 662–668 (1943). (Dutch. German, English and French summaries) [MF 15784]

This is a continuation of part IV [see the preceding review]. There two methods were indicated for obtaining contact transformations of E_m in a manifold X_n ($m+1<n$), which are not extended point transformations. The present communication considers the second method. For this purpose two $\mathfrak{S}_d^m$-fields in X_n are considered, and a contact transformation is defined as a (1, 1)-correspondence, mapping all E_m of the first $\mathfrak{S}_d^m$-field on those of the second field, which preserves coincidence of situation. Such a transformation may also be interpreted as a point transformation of two X_{n+d} which map two E_{m+d}-fields in three-space on each other. By means of canonical forms of E_2-fields in X_4 the case $m=1$, $n=3$, $d=1$ is discussed. A counter-example is given refuting the last theorem of F. Engel's dissertation [Math. Ann. 23, 1–44 (1884)].
D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. VI. Simple proof of the principal theorem for the general $\mathfrak{S}_d^{n-1}$-field. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 17–22 (1943). (Dutch. German, English and French summaries) [MF 15772]

[For the first five papers of this series see Nederl. Akad. Wetensch., Proc. 43, 18–31, 179–188, 453–462, 674–686 (1940); 45, 624–629 (1942); these Rev. 1, 145; 2, 54; 6, 67.] In this paper a simplified proof is given of the principal theorem of the general $\mathfrak{S}_d^{n-1}$-field [see part V], which does not use the theorems of Cartan-Kähler. The proof is given with the aid of a correspondence between the field of $(n-1)$-directions of the X_n resulting by selection from the $(n-1)$-directions of the $\mathfrak{S}_d^{n-1}$-field, and a system of ∞^1 X_n in an X_{2n-m}. The theorem can then be reduced to a problem solved by É. Goursat [Leçons sur le Problème de Pfaff, Paris, 1922, p. 165].
D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. VII. The principal theorem for the general $\mathfrak{B}_{n-m}$-field. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 138–145 (1943). (Dutch. German, English and French summaries) [MF 15775]

[Cf. the preceding review.] A general $\mathfrak{B}_{n-m}$-field is a field of sets of ∞^{n-m} covariant vectors at each point ξ^κ of an X_n. Such a $\mathfrak{B}_{n-m}$-field is given by means of m equations

$$B_b(\xi^\kappa, w_\lambda) = 0, \qquad \kappa, \lambda = 1, \cdots, n;\ b = m+1, \cdots, n,$$

the matrix of the $\partial B_b/\partial w_\kappa = B_b^\kappa$ having the highest rank m. The w_λ can be expressed as functions of the ξ^κ and of $n-m$

parameters w_y, $y=m+1, \cdots, n$: $w_\lambda = C_\lambda(\xi^\kappa, w_y)$. From B_b and C_λ we derive the tensors $K_b = B_b{}^\lambda C_\lambda$, $K_{cb} = 2B^\mu_{[c}\bar{\partial}_{|\mu|}B_{b]}$ ($\bar{\partial}$ meaning that w_λ is kept constant). The rank of the system K_b, K_{cb} is the class K^* of the $\mathfrak{B}_{n-m}$-field. The principal theorem of integrability is now proved for the case of a $\mathfrak{B}_{n-m}$-field of class K^*. It is shown that the solutions of class K of a $\mathfrak{B}_{n-m}$ can also be obtained by adding $n-m$ equations to the m equations of the field, so that a system of n equations of the class K arises from which the w_λ can be found. Here K satisfies the conditions

$$K^* \leqq K \leqq K^* + 2(n-m), \qquad K \leqq n,$$

or

$$K^* + 1 \leqq K \leqq K^* + 2(n-m), \qquad K \leqq n,$$

the first if the B_b are nonhomogeneous in w_λ, the second if they are homogeneous. For $K^* = K = 1$ we obtain the second method in Jacobi's solution for the nonhomogeneous case.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. VIII. Normal forms for systems of Pfaffian equations. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 197–200 (1943). (Dutch. German, English and French summaries) [MF 15777]

[Cf. the two preceding reviews.] A normal form is deduced for q Pfaffian equations in n variables, a deduction which so far has only been achieved in a few cases, to be found in Goursat's book on the problem of Pfaff. If $n-q$ is odd the normal form consists of q equations of class $n-q$; if $n-q$ is even it consists of $q-1$ equations of class $n-q+1$ and one of class $2(n-q)-1$. In the case when $m=n-q$ is even the normal form is as follows:

$$\begin{gathered} d\overset{1,0}{s} + \overset{1,1}{z}\, d\overset{1,1}{s} + \cdots + \overset{1,l}{z}\, d\overset{1,l}{s} = 0, \\ \cdots \qquad \cdots \\ d\overset{q,0}{s} + \overset{q,1}{z}\, d\overset{q,1}{s} + \cdots + \overset{q,l}{z}\, d\overset{q,l}{s} = 0, \end{gathered}$$

where $l = \frac{1}{2}(m-1)$. Some relations between the principal arithmetic invariants of the field are given.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. IX. Properties of the general vector distribution. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 415–420 (1943). (Dutch. German, English and French summaries) [MF 15779]

[Cf. the three preceding reviews.] A vector-$\mathfrak{M}_{2n-m}$ is a system of ∞^{2n-m} vector elements in an X_n defined by a system of m equations

$$B_b(\xi^\kappa, w_\lambda) = 0, \qquad \kappa, \lambda = 1, \cdots, n;\ b = 1, \cdots, m,$$

in which the rank of the matrix of the derivatives of the B_b with respect to ξ^κ and w_λ is m, the rank of $B_b{}^\kappa = \partial B_b/\partial w_\kappa$ with respect to w_λ is r, $r \leqq m$ and $r \leqq n$, and the rank of the matrix of the derivatives of the B_b with respect to the ξ^κ is r', $r' \leqq m$ and $r' \leqq n$. We see by comparison with part VII [see the second preceding review] that for $r = m \leqq n$ the vector $\mathfrak{M}_{2n-m}$ is a $\mathfrak{B}_{n-m}$-field. If $r < m$ the vector $\mathfrak{M}_{2n-m}$ is a $\mathfrak{B}_{n-r}$-field on an X_{n-m+r}. Three theorems are announced, the first stating that every vector-$\mathfrak{M}_{2n-m}$ ($m \leqq n$, $r < m$) can be transformed into a vector-$\mathfrak{M}_{2n-m}$ with $r = m$ by a homogeneous contact transformation, another giving normal forms to a vector-$\mathfrak{M}_{2n-m}$, a third stating the principal theorem of integrability. In the first theorem the vector-$\mathfrak{M}_{2n-m}$ should be nonhomogeneous for $m = n$ [see the following paper].

D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. X. Proofs of the theorems for the vector $\mathfrak{M}_{2n-m}$. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 571–574 (1943). (Dutch. German, English and French summaries) [MF 15781]

In this paper two of the theorems announced in the paper reviewed above are demonstrated, including the principal theorem of integrability.

D. J. Struik.

Schouten, J. A., and van der Kulk, W. Contributions to the theory of systems of Pfaffian equations. XI. Derivation of the canonical form for vector distributions. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 646–653 (1943). (Dutch. German, English and French summaries) [MF 15783]

[Cf. the five preceding reviews.] Here the theorem is demonstrated that a vector-$\mathfrak{M}_{2n-m}$ can be transformed into another vector-$\mathfrak{M}_{2n-m}$ by a homogeneous contact transformation, if and only if the $\mathfrak{M}_{2n-m}$ have the same class and index. This theorem is used to deduce the normal forms enunciated in part IX.

D. J. Struik.

Pfeiffer, G. V. New ways of researches in Pfaff's problem and integration of partial equations. Praci Sičnevoï Sesiï Akad. Nauk URSR. Dopovidi Viddilu Fiz.-Him. Mat. Nauk 2, 202–206 (1944). (Ukrainian. Russian and English summaries)

A summary of work dealing with the symbolic forms in $n+k$ variables $x_{\sigma_1} \cdots x_{\sigma_n}$,

$$\begin{aligned} \Omega = M dx_1 dx_2 \cdots dx_n + \sum_{\sigma_1 \tau_1}^{kn} M^{\sigma_1}_{\tau_1} dx_1 \cdots dx_{\tau_1 - 1} dz_{\sigma_1} \quad {}_{1+1} \cdots dx_n \\ + \sum_{\sigma_1 \sigma_2 \tau_1 \tau_2}^{kn} M^{\sigma_1 \sigma_2}_{\tau_1 \tau_2} dx_1 \cdots dx_{\tau_1 - 1} dz_{\sigma_1} dx_{\tau_1 + 1} \cdots \\ \times dx_{\tau_2 - 1} dz_{\sigma_2} dx_{\tau_2 + 1} \cdots dx_n + \cdots \end{aligned}$$

and the corresponding equations

$$0 = J = M + \sum_{\sigma_1 \tau_1}^{kn} M^{\sigma_1}_{\tau_1} \frac{\partial z_{\sigma_1}}{\partial x_{\tau_1}} + \sum_{\sigma_1 \sigma_2 \tau_1 \tau_2}^{kn} M^{\sigma_1 \sigma_2}_{\tau_1 \tau_2} \frac{\partial(z_{\sigma_1}, z_{\sigma_2})}{\partial(x_{\tau_1}, x_{\tau_2})} + \cdots$$

which are linear in Jacobians. Results of Hamburger and Goursat can be found in new ways and generalized to the case of systems of equations linear in Jacobians.

D. J. Struik (Cambridge, Mass.).

Lopatinsky, J. Linear differential operators. Rec. Math. [Mat. Sbornik] N.S. 17(59), 267–288 (1945). (Russian. English summary) [MF 16671]

The author studies linear (partial) differential equations from a formal algebraic point of view. His treatment of this special case of the algebraic theory of algebraic differential equations yields a well-rounded ideal theory of linear differential operators; in many respects it differs essentially from the treatment due to Ritt (for instance, ideals and sums of integral manifolds are defined differently). If $C(x)$ denotes the field of formal meromorphic functions in n variables x_i, the ring π of linear differential operators is defined as the $C(x)$-module whose elements are the finite linear combinations (coefficients in $C(x)$) of the basic operators

$$\partial^{k_1+k_2+\cdots+k_n}/\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n},$$

$0 \leqq \sum k_i < \infty$. Addition and multiplication in π are defined in the usual operational sense. The elements of π are thought of as operators in the ring $C(\xi)[x-\xi]$ of formal power series in $x_1-\xi_1, \cdots, x_n-\xi_n$, over $C(\xi)$. In applications, the point ξ will be any point of the domain of regularity of an analytic function, or of a set of such functions, and therefore from a formal point of view the ξ_i are regarded as indeterminates. The field $C(x)$ is considered as a subfield of $C(\xi)[x-\xi]$, in virtue of the formal expansion into a power series which exists at the point ξ for any function in $C(x)$. For any positive integer k the ordered k-tuples $(\alpha_1, \cdots, \alpha_k)$, $\alpha_i \epsilon \pi$, form a left module over π. The submodules of this module are called k-modules (the 1-modules are the left ideals in π). The paper centers around the study of k-modules and their integral manifolds. An integral of a given k-module M is a set of k elements $f_1, \cdots, f_k$ of $C(\xi)[x-\xi]$ such that $\alpha_1 f_1 + \cdots + \alpha_k f_k = 0$ for any element $(\alpha_1, \cdots, \alpha_k)$ in M. The integral manifold of M is the totality of the integrals of M. It is proved that any module has a finite basis; from this follow the usual theorems concerning the representation of a given k-module as an intersection of irreducible k-modules. Most of the results on integral manifolds are based on the following auxiliary theorem: a necessary and sufficient condition that a given system of linear equations $\alpha_{i1}u_1 + \cdots + \alpha_{ik}u_k = g_i$, $\alpha_{ij} \epsilon \pi$, $g_i \epsilon C(\xi)[x-\xi]$; $i=1, \cdots, l$ have a solution in $C(\xi)[x-\xi]$ is that, for any set of operators $\lambda_1, \cdots, \lambda_l$ in π which satisfy the relations $\lambda_1\alpha_{1j} + \cdots + \lambda_l\alpha_{lj} = 0$, $j=1, \cdots, k$, the relation $\lambda_1 g_1 + \cdots + \lambda_l g_l = 0$ is also satisfied. By means of this theorem the author derives for the integral manifolds of (finite or infinite) sums and (finite) intersections of k-modules distributive laws similar to those which hold in the theory of polynomial ideals and their zero manifolds. However, the sum of integral manifolds is intended in the algebraic, not in the set-theoretic sense. Moreover, it turns out that distinct modules have distinct integral manifolds. With a natural definition of convergent sequences in $C(\xi)[x-\xi]$ it is found that the integral manifold of an infinite intersection of k-modules M_i is the closure of the sum of the integral manifolds of the component modules M_i. Further results in this paper concern matrices of differential operators, the rank of such matrices and applications to the solution of a system of linear differential equations, both in the formal sense and in the ordinary sense of analysis.

O. Zariski (Urbana, Ill.).

De Donder, Th. Une manière simplifiée pour résoudre le problème de Cauchy dans le cas d'un système d'équations linéaires aux dérivées partielles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 8–10 (1945).

The author calls attention to the trivial but important fact that the differential equations in a linear Cauchy problem can always be modified so that the given Cauchy problem is equivalent to another in which the Cauchy data are identically zero. This is true even of nonlinear systems.

D. C. Lewis (College Park, Md.).

Meltzer, L. A. On the correct statement of Goursat's problem. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 59–104 (1946). (Russian. English summary) [MF 16677]

A systematic study is given of conditions under which the problem of Goursat for the system

$$(1) \qquad \sum_{i=1}^{l+m} f_{ji}(x, y)\partial u_i/\partial x + \sum_{i=1}^{l+m} \phi_{ji}(x, y)\partial u_i/\partial y = \Phi_j,$$

$j=1, \cdots, l+m$; Φ_j depending on $x, y, u_1, \cdots, u_{l+m}$, is correctly stated in a bounded domain D, situated in the first quadrant so that the boundary of D contains the segments $0 \leqq x \leqq a$, $0 \leqq y \leqq b$; that is, conditions are given under which one has the following. (A) For every set of continuous and continuously differentiable bounded $\psi_\lambda(x)$ ($\lambda \leqq l$ on $(0, a)$), $\psi_\mu(y)$ ($l < \mu \leqq l+m$ on $(0, b)$) there exists just one set u_i (the u_i and their first order derivatives continuous in D), satisfying (1) and the relations $u_\lambda = \psi_\lambda(x)$ for $y=0$ (x on $(0, a)$; $\lambda \leqq l$), $u_\mu = \psi_\mu(y)$ for $x=0$ (y on $(0, b)$; $l < \mu \leqq l+m$). (B) Arbitrarily small variations in ψ_i, $\psi_i^{(1)}$ result in arbitrarily small variations in the u_i. *W. J. Trjitzinsky* (Urbana, Ill.).

Savarensky, E. F. The unrestricted applicability of S. Tchaplygin's theorem on differential inequalities to linear equations with partial derivatives of the first order. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 259–261 (1946).

It is shown that if $z(x, y)$ is a solution of the equation $L(z) \equiv z_y + A(x, yz)z_x - R(x, yz) = 0$ which passes through an initial curve $C(y=y_0, z=\phi(x))$, then if $v(x, y)$ also passes through C and is such that $L(v) > 0$ on $x_0 < x < x_1$, $y_0 < y < y_1$, then $v(x, y) > z(x, y)$ on this rectangle.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Drach, Jules. Sur les lignes de flux qui sont lignes de tourbillon. C. R. Acad. Sci. Paris **223**, 441–444 (1946).

The author outlines a reduction of the system (S_0): $v_x - u_y = \lambda w, \cdots$, to four linear equations of the fourth order in u and applies his results to various special cases, including the case where $\lambda = 1$ and the case where u and v are independent of z. *C. C. Torrance* (Annapolis, Md.).

Gosse. Quelques équations $0=f(x, y, z, p, q)$ intégrables par la méthode de Darboux. Ann. Univ. Lyon. Sect. A. (3) **4**, 83–84 (1941).

Vicente Gonçalves, J. Quelques problèmes classiques de valeurs propres. Revista Fac. Ci. Univ. Coimbra **10**, 195–246 (1942).

Expository article, intended as an introduction to quantum mechanics.

Chernoff, Herman. Complex solutions of partial differential equations. Amer. J. Math. **68**, 455–478 (1946).

L'auteur étudie l'équation aux dérivées partielles

$$(1) \qquad \Delta u + A(x, y)\partial u/\partial x + B(x, y)\partial u/\partial y + C(x, y)u = 0$$

selon la méthode de Bergman [Trans. Amer. Math. Soc. **57**, 299–331 (1945); ces Rev. **7**, 16] qui fournit les intégrales comme parties réelles des fonctions

$$u(z, \bar{z}) = \int_{-1}^{1} E(z, \bar{z}, t) f(\tfrac{1}{2}z(1+t^2))(1-t^2)^{-\frac{1}{2}} dt,$$

où E est fixé en correspondance avec (1) et $f(\zeta)$ une fonction analytique variable régulière à l'origine. Il s'agit ici de transposer des résultats classiques sur les fonctions méromorphes en étudiant et comparant les allures pour $r \to \infty$ de

$$(2\pi)^{-1} \int_0^{2\pi} \log^+ |u(re^{i\varphi}, re^{-i\varphi})| d\varphi,$$

$$(2\pi i)^{-1} \int_{|z|=r} d \log (u(z, \bar{z}) - b),$$

et cela suggère une sorte d'analogue au théorème de Picard sur les fonctions entières. *M. Brelot* (Grenoble).

Mukherjee, Santi Ram. Solutions of some differential equations arising in problems of varying viscosity in hydrodynamics. Proc. Nat. Acad. Sci. India. Sect. A. 12, 46–65 (1942).

The author solves, in effect, the following partial differential equations by the method of separation of variables:

$$s\nabla^2 V=kV,\quad r\nabla^2 V=kV,\quad r^2\nabla^2 V=kV,$$
$$(a+br)\nabla^2 V=kV,\quad (a+bx)\nabla^2 V+cV=0.$$

E. T. Copson (Dundee).

Wassermann, Gerhard. Sur quelques problèmes de fonctions propres relatifs à l'hexagone régulier. C. R. Acad. Sci. Paris 223, 537–539 (1946).

The author discusses briefly the equation $\psi_{xx}+\psi_{yy}+k\psi=0$, where either ψ or its normal derivative vanishes on the perimeter of a regular hexagon. The solution is essentially that obtained by separation of variables. *H. Pollard.*

Frankl, F. On the theory of the equation $y\partial^2z/\partial x^2+\partial^2z/\partial y^2=0$. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 135–166 (1946). (Russian. English summary)

Soient D un domaine simplement connexe du plan Oxy, D' sa frontière; D' se compose du segment OA ($y=0$, $0\leqq x\leqq 1$) et d'un arc de courbe L, reliant O à A, situé à l'intérieur du demi-plan $y>0$ (extrémités exclues), doué d'une tangente continue et d'une courbure partout bornée; les tangentes à L en O et A seront perpendiculaires à Ox. Soient s l'arc de L compté à partir de O ou de A; $f(s)$ une fonction donnée, définie et continue sur L; $f'(s)$ existe et est bornée, mais peut présenter un nombre fini de points de discontinuité; $f''(s)$ existe, est bornée le long de L, sauf, éventuellement, en O et A, où $f''(s)$ se comporte alors comme y^{-a} $(0<a<1)$. Soient $\tau(x)$ et $\nu(x)$ deux fonctions données, définies et continues pour $0<x<1$; $\tau'(x)$ existe pour $0\leqq x\leqq 1$, est bornée et peut présenter un nombre fini de discontinuités; $\nu(x)$ est différentiable pour $0<x<1$ et se comporte comme $O(1)(1-x)^{-\frac{1}{3}}x^{-\frac{1}{3}}$ en O et A.

Considérons alors l'équation de Darboux-Tricomi

$$(1)\qquad y\partial^2z/\partial x^2+\partial^2z/\partial y^2=0,$$

elliptique dans $y\geqq 0$; la transformation $X=x$, $Y=\frac{2}{3}y^{\frac{3}{2}}$ la change en

$$(2)\qquad \Delta z+\tfrac{1}{3}Y^{-1}\partial z/\partial Y=0$$

pour $y\geqq 0$. Les plans Oxy et OXY seront utilisés conjointement. Relativement à (1) et à D, l'auteur étudie deux problèmes. (A) (Problème de Dirichlet). Trouver une solution $z(x,y)$ de (1), régulière dans D, se réduisant à $f(s)$ sur L, à $\tau(x)$ sur OA, bornée dans le voisinage des discontinuités de $f'(s)$ et $\tau'(x)$. (B) Trouver une solution $z(x,y)$ de (1), régulière dans D, se réduisant à $f(s)$ sur L, et telle que $z_y'=\nu(x)$ sur OA; z doit rester bornée dans le voisinage des points intérieurs de OA.

A noter: l'unicité de la solution dans le cas (B) n'est assurée que moyennant la condition supplémentaire, écrite dans le plan OXY: si $c(\epsilon)$ est l'arc du cercle $X^2+Y^2=\epsilon^2$ (ou $(1-X)^2+Y^2=\epsilon^2$) intérieur à D, n la normale à $c(\epsilon)$, S l'arc de $c(\epsilon)$, on doit avoir

$$\lim_{\epsilon\to 0}\int_{c(\epsilon)} z(dz/dn)Y^{\frac{1}{3}}dS=0.$$

Le problème (A) a été étudié par Tricomi, le problème (B) par Tricomi et Gellerstedt; les méthodes de l'auteur sont plus simples que celles de Tricomi, les hypothèses faites relativement à L plus larges que celles de Gellerstedt.

L'auteur opère d'abord la réduction des problèmes (A) et (B) aux cas particuliers où $\tau=0$, $\nu=0$. Il construit ensuite, pour l'équation (2), deux solutions fondamentales, définies pour $Y\geqq 0$ et possédant dans ce domaine une singularité logarithmique. L'inconnue z des deux problèmes peut alors être mise sous forme d'un potentiel de double couche de densité $\mu(s)$ étalée sur L; $\mu(s)$ se trouve être solution d'une équation de Fredholm de deuxième espèce dont le noyau est construit à partir des solutions fondamentales. L'auteur montre que la théorie de Fredholm s'applique à cette équation. Dans l'annexe à son mémoire, l'auteur rectifie quelques assertions inexactes de Tricomi. *J. Kravtchenko.*

Chabate, B. Sur les solutions généralisées des systèmes elliptiques linéaires. Rec. Math. [Mat. Sbornik] N.S. 17(59), 193–210 (1945). (Russian. French summary) [MF 16668]

Envisageons le système elliptique d'équations aux dérivées partielles:

$$(1)\qquad a_1u_x+a_2u_y-v_y=0,\quad a_3u_x+a_4u_y-v_x=0,$$

$a_1a_4-(a_2+a_3)^2/4>0$. L'auteur étudie les couples de fonctions $u(z)$ et $v(z)$, définies dans le domaine simple D du plan $z=x+iy$ et qui y possèdent les propriétés suivantes. (1) Les fonctions u et v possèdent presque partout une différentielle totale au sens de Stolz et vérifient presque partout (1). (2) Les dérivées partielles de premier ordre de u et v sont de carré sommable. (3) Soient les segments $x=x_0$ ou $y=y_0$ intérieurs à D; sauf pour un ensemble dénombrable de ces segments, $f(z)=u+iv$ transforme tout ensemble parfait, de mesure linéaire nulle, situé sur un tel segment, en un ensemble de mesure nulle. (4) La représentation $w=f(z)$ sur le plan w est continue, ouverte, ne transformant pas un continu distinct d'un point en un point.

Cela étant, l'auteur construit une formule de Green pour exprimer u et v; ce mode de représentation des inconnues lui permet de relier les propriétés de continuité de u et v à celles des $a_i(z)$. En utilisant et en adaptant convenablement les raisonnements de E. Hopf [Math. Z. 34, 194–233 (1931)], l'auteur parvient à l'énoncé suivant: si les $a_i(z)$ admettent dans D des dérivées partielles d'ordre m, vérifiant dans D une condition de Hölder d'exposant δ, les dérivées partielles d'ordre $m+1$ de toute solution u, v de (1) existent et vérifient une condition de Hölder d'exposant δ.

L'auteur définit ensuite une classe de correspondances quasi-conformes $w=u+iv$, généralisant celle de M. Lavrentieff [même Rec. 42, 407–423 (1937)]; il vérifie que les fonctions u et v sont alors solutions de (1); la réciproque est vraie. On indique enfin quelques propriétés fonctionnelles de ces correspondances: limitation des coefficients de déformations, critère de compacité des dérivées partielles de u et v.

J. Kravtchenko (Grenoble).

Kondrachoff, V. I. Sur le problème limite dans un domaine à contour dégénéré pour certaines équations opératoires non linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 415–418 (1946).

If a domain in Euclidean space has a smooth boundary which is a sum $\sum_k S_{n-k}$, where each S_{n-k} is an $(n-k)$-dimensional manifold, then certain types of partial differential equations can be solved if a boundary value φ_{n-k} and some of its derivatives for some k are prescribed. The author

affirms, in particular, the possibility of treating certain non-linear equations of the type $\sum_{r,p}\Delta_r'(\Delta_r''\varphi)^{2p-1}=0$, where Δ_r', Δ_r'' are partial differential monomials of the type $\partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$ and the superscript $2p-1$ indicates taking ordinary powers. The details are too condensed for further comment. *S. Bochner* (Cambridge, Mass.).

Chandrasekhar, S. A new type of boundary-value problem in hyperbolic equations. Proc. Cambridge Philos. Soc. **42**, 250–260 (1946).

The problem in question is as follows: to find the solution of

$$(*)\qquad \partial^2 f/\partial x^2-\partial^2 f/\partial y^2+f=0$$

given that (i) $f=F(x)$, $\partial f/\partial y=G(x)$ for $y=0$, $0\leqq x\leqq l$, (ii) $\partial f/\partial x-\partial f/\partial y=\phi(y)$ for $x=0$, $y\geqq 0$, (iii) $f=\psi(y)$ for $x=l$, $y\geqq 0$. In two previous papers [Rev. Modern Physics **17**, 138–156 (1945); Astrophys. J. **102**, 402–428 (1945); these Rev. **7**, 177, 304], the author has shown that the problem can be solved in principle by an extension of Riemann's method, but that there is the difficulty of having to solve a succession of Volterra integral equations. He now gives an alternative solution which involves solving only one integral equation (which does not involve the boundary values) and reduces the problem to one involving only quadratures.

The new idea is to construct a solution $C(x, y; x_1, y_1)$ of $(*)$ which takes the value unity on the characteristic $x+y=x_1+y_1$, $x\geqq x_1$ and on the line $x=x_1$, $y\leqq y_1$. That this solution is

$$C=\cos(x-x_1)+\tfrac{1}{2}(x-x_1)\int_0^{\pi} J_0\{(x-x_1)\sin\theta\}$$
$$\times \mathrm{Ii}_1\{y_1-y-(x-x_1)\cos\theta\}\sin\theta d\theta$$

is proved by using Riemann's method and solving a Volterra integral equation by means of the Laplace transform. The function $C'(x, y; x_1, y_1)\equiv C(2x_1-x, y_1; x_1, y_1)$ takes the value unity on the characteristic $x-y=x_1-y_1$, $x\leqq x_1$ and the line $x=x_1$, $y\leqq y_1$. It is also necessary to introduce the function $\Gamma(x, y; x_1, y_1)\equiv\partial C/\partial x-\partial C/\partial y$.

The solution of the problem in the square $0\leqq x\leqq l$, $0\leqq y\leqq l$ is straightforward if we can find f on $x=0$, $0\leqq y\leqq l$ and $\partial f/\partial x$ on $x=l$, $0\leqq y\leqq l$; the continuation of the solution into the next square $0\leqq x\leqq l$, $l\leqq y\leqq 2l$ will then follow similarly. To find f on $x=0$, the author applies Green's identity

$$(**)\qquad \int(v\partial f/\partial y-f\partial v/\partial y)dx+(v\partial f/\partial x-f\partial v/\partial x)dy=0$$

with $v\equiv\Gamma(x, y; 0, \eta)$ to the triangle with vertices $(0, 0)$, $(\eta, 0)$, $(0, \eta)$. It follows that

$$\int_0^{\eta} f(0, y)dy=\mathrm{Ii}_1(\eta)F(0)-\int_0^{\eta}\Gamma(0, y; 0, \eta)\phi(y)dy$$
$$-\int_0^{\eta}\{F(x)(\partial\Gamma/\partial y)_{y=0}-\Gamma(x, 0; 0, \eta)G(x)\}dx.$$

Similarly, by applying $(**)$ with $v\equiv C'(x, y; l, \eta)$ to the triangle with vertices $(l-\eta, 0)$, $(l, 0)$, (l, η), it is found that

$$\int_0^{\eta}\{\partial f/\partial x\}_{x=l}dy=\psi(\eta)-F(l-\eta)-\int_0^{\eta}\psi(y)\mathrm{Ii}_1(\eta-y)dy$$
$$+\int_{l-}^{l}\{F(x)(\partial C'/\partial y)_{y=0}-C'(x, 0; l, \eta)G(x)\}dx.$$

By differentiation, the boundary values of f and $\partial f/\partial x$ are obtained in terms of known functions by quadratures.

With a view to practical applications the author gives a table of values of the Bessel integral

$$\mathrm{Ii}_1(x)=\int_0^{x} t^{-1}I_1(t)dt,$$

correct to six significant figures for the range 0(.1)5(.05)10-(.1)20. *E. T. Copson* (Dundee).

Soboleff, S. L. Sur la presque périodicité des solutions de l'équation des ondes. I. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 542–545 (1945). [MF 16645]

Soboleff, S. L. Sur la presque périodicité des solutions de l'équation des ondes. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 618–620 (1945). [MF 16640]

Soboleff, S. L. Sur la presque périodicité des solutions de l'équation des ondes. III. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 12–15 (1945). [MF 16638]

After a theorem of Muckenhoupt for $n=1$ [J. Math. Phys. Mass. Inst. Tech. **8**, 163–199 (1929)] it was proved by the reviewer [Acta Math. **62**, 227–237 (1934)] that a solution $u=u(x_1, \cdots, x_n; t)$ of the wave equation $(*)$ $\Delta u-\partial^2 u/\partial t^2=0$, if compact as a function of t, is automatically almost periodic in t. The operator Δu in this statement is a fairly general elliptic operator over a suitable Hilbert space of functions of $(x_1, \cdots, x_n)$ in a domain; the "compactness" and "almost periodicity" refer to the function $u(x; t)$ as an abstract-valued function in t whose values are elements of the Hilbert space. In a paper by the reviewer and von Neumann [Ann. of Math. (2) **36**, 255–291 (1935)] this was generalized to solutions of more general equations which are linear in t.

In the present notes the author is concerned in the case of equations $(*)$ with drawing up assumptions under which the prerequisite compactness will be verified in order to be able to draw the conclusion that the trajectories are almost periodic (in the average). In note I he proves this compactness for the ordinary Laplacian simply under the alternate boundary conditions $u|_S=0$ or $\partial u/\partial n|_S=0$ with certain smoothness requirements on the boundary and differentiability conditions on $u(x; t)$. He obtains this conclusion by utilizing, in addition to the familiar energy integral $\int(\sum(\partial u/\partial x_i)^2+u^2)d\Omega$, which had been the only one used before, a new type of energy integral which involves partial derivatives of the second order, and whose constancy permits the conclusion (partly based on results of W. Kondrachov) that the partial derivatives of the first order in x are also compact. It is unlikely that the integral has not been noticed before, but its use in this type of problem seems to be novel.

In note II the author generalizes his conclusions from the Laplacian in rectilinear coordinates to one in curvilinear coordinates. In note III he returns again to the classical operator $\Box u=\sum(\partial^2 u/\partial x_i^2)-\partial^2 u/\partial t^2$; however, he admits "generalized" solutions of $\Box u=0$ for which no partial derivatives in x need exist. Such generalized solutions are defined by the adjoint equation $(u, \Box\phi)=0$, in which ϕ belongs to a large class of differentiable functions. The author proves again that the boundary condition $u|_S=0$ insures compactness and thus almost periodicity of the trajectory. The latter type of "generalized" solution by means of adjoint equations has also been treated in the meantime by the reviewer [Ann. of Math. (2) **47**, 202–212 (1946); these Rev. **7**, 446]. *S. Bochner.*

Petrowsky, I. On the diffusion of waves and the lacunas for hyperbolic equations. Rec. Math. [Mat. Sbornik] N.S. 17(59), 289–370 (1945). (English. Russian summary) [MF 16672]

The following summary is taken from the author's introduction [cf. C. R. (Doklady) Acad. Sci. URSS 38, 151–153 (1943); Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 8, 101–106 (1944); these Rev. 5, 8; 6, 229].

In the present paper we consider hyperbolic systems [cf. Petrowsky, same Rec. N.S. 2(44), 815–868 (1937)] of the form

$$(3)\qquad \partial^{n_i}u_i/\partial t^{n_i}=F_i(t, x_1, \cdots, x_p; u_1, \cdots, u_N, \cdots),\quad i=1, \cdots, N,$$

where the derivatives of the form

$$\frac{\partial^k u_j}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_p^{k_p}},\qquad \sum k_s=n_j;\ k_0<n_j;\ j=1, \cdots, N,$$

are not indicated explicitly in F_i. The functions F_i are supposed to have derivatives of sufficiently high order with respect to all their variables [cf. the paper cited above]. We say that there is a lacuna L for the system (3) in the neighborhood of the initial data

$$\partial^k u_i/\partial t^k=\bar{u}_i^{(k)}(x_1, \cdots, x_p),\quad i=1, \cdots, N; k=0, 1, \cdots, n_i-1,$$

for a point $P(t^*, x_1^*, \cdots, x_p^*)$ if the values at P of the functions $u_1, \cdots, u_N$ satisfying this system do not depend on the values of the initial data at $t=t_0=$constant on the domain L lying in the base of the characteristic cone with vertex at P, that is, $u_i(t^*, x_1^*, \cdots, x_p^*)$ do not vary when $\bar{u}_i^{(k)}(x_1, \cdots, x_p)$ are replaced by functions which, with their derivatives up to a certain order, differ but slightly from $\bar{u}_i^{(k)}$ everywhere and coincide with $\bar{u}_i^{(k)}$ outside a domain L^* lying, together with its boundary, in L. We shall consider only the domains L cut out by the characteristic cone having its vertex at P. This cone is constructed for the solution determined by the initial data $\bar{u}_i^{(k)}(x_1, \cdots, x_p)$ and therefore, in the case of a nonlinear system, this cone and the lacuna L depend on the initial data. In the case of linear systems there is no such dependence.

In chapter II we investigate the lacunas for linear systems in the case $p=1$. In chapter I we establish a relation between the existence of lacunas for general nonlinear hyperbolic systems and that for the corresponding linear systems with constant coefficients containing only the highest derivatives of all the functions. For one hyperbolic equation of such a kind the formulae for the solution of Cauchy's problem were given by Herglotz for $p>1$. Since these formulae are to be modified and generalized for our purposes, we give in chapter III a detailed proof in the required form without unnecessary restrictions.

Using the formulae obtained, in chapter V we give necessary and sufficient conditions for the existence of stable lacunas for a linear hyperbolic equation with constant coefficients containing the highest derivatives only. A lacuna L for such an equation is called stable if it is not destroyed by sufficiently small variations of the coefficients. At the beginning of chapter V we suppose that the characteristic equation of the hyperbolic equation represents a nondegenerate algebraic surface without singular points. The problem of lacunas is studied for the case where this surface is degenerate. The existence of lacunas is discussed for a system of equations. The results are applied to some equations of mathematical physics.

In chapter IV some theorems that are necessary for chapter V, regarding integrals over algebraic surfaces, are proved.

As is known, for $t^*>0$ and any odd $p>1$ the value of the solution u of the equation (1) $u_{tt}=\sum_{k=1}^p u_{x_kx_k}$ at the point $(t^*, x_1^*, \cdots, x_p^*)$ depends only on the values of u at $t=0$ (initial values) and of the derivative of u with respect to t on the sphere (2) $\sum_{k=1}^p(x_k-x_k^*)^2=(t-t^*)^2$ that belongs to the plane $t=0$ and is the intersection of the latter with the characteristic cone for the equation (1) having its vertex at the point $(t^*, x_1^*, \cdots, x_p^*)$. For p even or $p=1$, the value of $u(t^*, x_1^*, \cdots, x_p^*)$ depends not only on the values of the initial data on the sphere (2), but also on their values within it.

This implies that in case p is odd the spherical wave produced in a small neighborhood of a point Q of the $(x_1, \cdots, x_p)$-space has the property that both its front and back edges are sharp. As to the case where p is even or $p=1$, only the front edge of such a wave is sharp, while the back edge is diffuse. In the first case it is said that there is no diffusion of waves for the equation (1), in the second case, that the diffusion of waves takes place [J. Hadamard, Le Problème de Cauchy, Paris, 1932, p. 238].

It may happen that for a hyperbolic equation or for a hyperbolic system the value of the solution of Cauchy's problem at the point $P(t, x_1, \cdots, x_p)$ does not depend on the values of the initial data on some of those domains into which the base of the characteristic cone with vertex at P is decomposed by this cone. This possibility is realized, for instance, for the fundamental equations of the theory of elasticity. We say that such domains are lacunas for the systems. *E. T. Copson* (Dundee).

Faedo, Sandro. Ulteriori contributi alla teoria del metodo variazionale. Ann. Scuola Norm. Super. Pisa (2) 12, 99–116 (1943). [MF 16769]

The author considers the variational method of Picone for a propagation problem of the sort which he has previously treated [same Ann. (2) 10, 139–152 (1941); these Rev. 3, 245]. He shows that the convergence in the mean of a sequence of functions occurring in the variational method follows from the hypothesis that there exists a solution of the given boundary problem and does not require further hypotheses on the differentiability of this solution as originally assumed by Picone [Rend. Accad. Sci. Fis. Mat. Napoli (4) 6, 217–235 (1936)]. This weakening of conditions is attained through the use of multiple Fourier series. In addition, there is a brief discussion of the application of the variational method to the solution of nonlinear partial differential equations. *W. T. Reid.*

Michal, Aristotle D. The vibrations of elastic strings as studies in geodesics. Actas Acad. Ci. Lima 9, 3–27 (1946).

According to the author, the paper can be considered as a first chapter in a program of geometrization of various physical domains along a quite different direction from the well-known geometrization of relativity and unified field theory. The discussion is limited to the transverse vibrations of a stretched string with fixed ends at $x=0$ and $x=l$ (that is, the one-dimensional wave equation). The general aim is to geometrize this problem in an infinite-dimensional Riemannian space R. If $u(x, t)$ is the displacement at position x on the string at time t, then for fixed t the function $u(x, t)$ on the range $0\leqq x\leqq l$ is regarded as a point in R. As the string moves in any arbitrary manner, the point describes a curve in R, t being a parameter for this curve. It is essential in the paper that the parameter t be changed

to arc length s, defined as below. Let F be the tension in the string, ρ the linear density and C a constant. Let $v(x)$ and $v(x)+\delta v(x)$ be two adjacent points in R. Then the Riemannian line element corresponding to this pair of points is defined by

$$ds^2=2\left[C-\tfrac{1}{2}F\int_0^l (dv(x)/dx)^2dx\right]\rho\int_0^l (\delta v(x_1))^2dx_1.$$

Consider now (artificial) motions of the string, subject to the conditions that the displacement shall vanish at the ends of the string and that the total energy shall have the assigned value C; furthermore, let all the motions begin and end with the same two shapes for the string. Each such motion has a length in R given by $\int ds$ and a motion having a stationary length is a geodesic in R. A central result of the paper is the theorem that those vibrations which satisfy the wave equation correspond to geodesics in R (generalization of Jacobi's principle of least action). Complete details of this proof are, however, not given. It is further shown that the harmonic vibrations of the string are closed geodesics in R, possessing certain orthogonality properties. The paper ends with definitions of Christoffel and curvature forms; it is stated that R is not of constant Riemannian curvature. *J. L. Synge* (Pittsburgh, Pa.).

Oseen, C. W. Le principe de Huygens et les équations de Maxwell pour le vacuum. Ark. Mat. Astr. Fys. **31A**, no. 4, 17 pp. (1944).

Let $\phi(x, y, z, t)$ be a solution of the wave equation $\nabla^2u=c^{-2}\partial^2u/\partial t^2$ which is regular inside and on a closed surface S. Let ψ be a second solution whose only singularity inside S is at (ξ, η, ζ). Then, if we surround the singularity by a small sphere σ of radius ϵ, we have

$$c^2\int_{t_0}^{\tau} dt\int_{S+\sigma}\left(\psi\frac{d\phi}{dN}-\phi\frac{d\psi}{dN}\right)dS+\left[\int_V(\dot\psi\phi-\dot\phi\psi)dV\right]_{t=t_0}^{\tau}=0,$$

where N is the unit normal vector drawn into the volume bounded by S and σ. Kirchhoff derived his formula for $\phi(\xi, \eta, \zeta, \tau)$ by taking $\psi=r^{-1}F(t+c^{-1}r-\tau)$, where r is distance from the singularity and F is what is now called Dirac's delta function. As there exists no function with the properties of Dirac's function, the author proposes to use instead the function $F(\theta)=\alpha^{-1}\pi^{-\frac{1}{2}}e^{-\theta^2/\alpha^2}$ which has the property that, under certain conditions, $\lim_{\alpha\to0}\int_a^b f(\theta)F(\theta-\phi)d\theta$ is equal to $f(\phi)$ or zero according as ϕ does or does not lie in the interval (a, b). This leads directly to the generalisation of Kirchhoff's formula which arises when $\tau\geqq t_0$, a fixed constant. An elaboration of the same idea is then applied to prove the corresponding formulae for the vectors **E** and **H** which satisfy Maxwell's equations in empty space.

From the rigorous standpoint, the new method is not entirely satisfactory since it involves inverting the order of integration in expressions such as

$$\int_{t_0}^{\tau} dt\int_\sigma\left\{\frac{1}{r}\frac{d\phi}{dN}-\phi\frac{dr^{-1}}{dN}\right\}F(t+c^{-1}r-\tau)dS$$

and then making α tend to zero within the sign of integration over σ. *E. T. Copson* (Dundee).

Drăganu, Mircea. Sur la résolution de l'équation différentielle des mouvements vibratoires d'une membrane rectangulaire par la méthode de la transformation multiple de Laplace. Mathematica, Timişoara 22, 206–207 (1946).

The author proposes to solve the equation (*) $z_{xx}+z_{yy}=z_{tt}$ subject to the conditions (a) $z(x, y, 0)$ and $z_t(x, y, 0)$ are prescribed functions of x and y and (b) $z(x, y, t)$ and its normal derivatives vanish on the boundary of a rectangle. Such boundary conditions are more than are required to fix the solution of (*). In the view of the absence of a number of steps in the derivation, it is not clear how the author obtains his solution. *A. E. Heins* (Pittsburgh, Pa.).

Michlin, S. Propagation des ondes dans les domaines limités par des courbes. Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog. no. 110, 50 pp. (1 plate) (1941). (Russian. French summary)

Nous considérons ici le problème aux limites relatif à l'équation d'onde $U_{xx}+U_{yy}-U_{tt}=0$. Nous supposons que la frontière du domaine est une courbe analytique sans points singuliers dont la courbure est différente de zéro. On suppose donnée sur la frontière soit la fonction inconnue soit sa dérivée normale. On suppose aussi données les fonctions U, U_t, pour $t=0$.

Nous introduisons des intégrales analogues aux potentiels. Ces intégrales satisfont à l'équation d'onde et s'annullent, ainsi que leurs dérivées du premier ordre, pour $t=0$. En utilisant ces potentiels nous réduisons notre problème à une équation intégrodifférentielle.

From the author's summary.

Masket, A. Victor. On the vibrations of a whirling wire. J. Acoust. Soc. Amer. 18, 216 (1946).

This letter to the editor contains a summary of a paper which is to appear elsewhere. The physical problem considered is reduced to solving the equation

$$\frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial v}{\partial y}\right\}=\frac{\partial^2 v}{\partial\tau^2}$$

under the following conditions: when $\tau=0$, $v=0$ and $\partial v/\partial\tau=0$ for $0\leqq y<\alpha l<1$; when $y=0$, $v=0$; when $y=\alpha l(<1)$, $v=B\sin\beta\tau$. Application of the Laplace transformation

$$\bar v(y, \lambda)=\int_0^\infty e^{-\lambda\tau}v(y, \tau)d\tau$$

gives

$$(*)\qquad \frac{d}{dy}\left\{(1-y^2)\frac{d\bar v}{dy}\right\}-\lambda^2\bar v=0$$

under the conditions $\bar v=0$ when $y=0$, $\bar v=B\beta/(\lambda^2+\beta^2)$ when $y=\alpha l$. It follows that

$$\bar v=\frac{B\beta P(y, \lambda^2)}{(\lambda^2+\beta^2)P(\alpha l, \lambda^2)},$$

$$P(y, \lambda^2)=y+\lambda^2+2y^3/3!+(\lambda^2+2)(\lambda^2+12)y^5/5!+\cdots.$$

The series P is convergent when $|y|<1$ for all λ and the equation $P(\alpha l, \lambda^2)=0$ has roots $\lambda=\pm i\lambda_n$, $n=1, 2, 3, \cdots$, where the numbers λ_n are positive.

It then follows by using the complex inversion formula for the Laplace transform and the calculus of residues that

$$v(y, \tau)=\frac{BP(y, -\beta^2)\sin\beta t}{P(\alpha l, -\beta^2)}+2B\beta\sum_1^\infty\frac{P(y_1-\lambda_n^2)\sin\lambda_n\tau}{(\beta^2-\lambda_n^2)W(\alpha l, \lambda_n)},$$

$$W(\alpha l, \lambda_n)=-i[dP(\alpha l, \lambda^2)/d\lambda]_{\lambda=i\lambda_n},$$

provided that β is not equal to one of the numbers λ_n. In the

exceptional case, double poles have to be considered. The equation (*) is Legendre's equation of complex order.

E. T. Copson (Dundee).

Cinquini-Cibrario, Maria. Sopra alcune questioni relative alle equazioni del tipo iperbolico non lineari. Ann. Mat. Pura Appl. (4) 23, 1–23 (1944). [MF 16609]

The paper contains refinements and additions to the author's previous paper on Goursat's problem [same Ann. (4) 21, 189–229 (1942); these Rev. 6, 4]. Where previously the initial curves γ_1, γ_2 and the initial data were assumed to be of class C^3, it is now seen to be sufficient to assume existence of third order derivatives satisfying a Lipschitz condition. In the case of a quasi-linear equation even existence of second order derivatives satisfying a Lipschitz condition is sufficient. Properties of characteristic curves and strips well-known from the analytic case are here extended to second order hyperbolic equations of class C^3. If a curve Γ on an integral surface S is characteristic with respect to S, then there exist infinitely many integral surfaces through Γ, with respect to which Γ is characteristic and which have contact of second order with S. Any curve along which two integral surfaces have contact of second order is characteristic with respect to those surfaces.

F. John (New York, N. Y.).

Brook, S. On Cauchy's problem for parabolic systems of differential equations. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 105–120 (1946). (Russian. English summary)

This paper extends a theorem of I. G. Petrowsky [Bull. Univ. État Moscou. Ser. Int. Sect. A. Math. Mécan. 1, no. 7 (1938)] on the correct statement of Cauchy's problem [J. Hadamard, Le Problème de Cauchy, Paris, 1932, pp. 238–241] for parabolic systems of linear differential equations to parabolic systems which are linear in the derivatives of highest order occurring in them but nonlinear in the lower order derivatives. The method consists in reducing the problem to a system of integro-differential equations which are solved by the method of successive approximations.

H. P. Thielman (Ames, Iowa).

Kronig, R., and van Gijn, G. On a problem of heat transfer between a moving medium and an extended solid. Physica 12, 118–128 (1946).

In an annealing furnace a stream of hot gas may pass along both sides of a cool metal plate. Here the velocity and width of the stream are assumed constant and other assumptions are introduced so that the temperatures of the plate and of the gas are functions of x and time, where x is the distance along the plate from the edge where the stream enters, measured in the direction of the stream. It is assumed that heat is transferred by convection only. A system of two simultaneous linear partial differential equations of the first order in the two temperature functions is set up, together with some boundary conditions. A solution is written and a chart is constructed from which certain temperature differences can be read.

R. V. Churchill.

McLachlan, N. W. Heat conduction in elliptical cylinder and an analogous electromagnetic problem. Philos. Mag. (7) 36, 600–609 (1945). [MF 16977]

Let $\theta(\xi, \eta, t)$ denote the temperatures in an infinitely long elliptic cylinder $\xi=\xi_0$, where ξ and η are elliptic coordinates; that is, $x=h\cosh\xi\cos\eta$, $y=h\sinh\xi\sin\eta$. The initial temperature of the cylinder is zero and the temperature of the surface is kept at a constant value θ_0. The author derives a formula for the temperature function θ in the form of a double infinite series whose terms are products of Mathieu functions of ξ and η and exponential functions of t. The coefficients in the series involve integrals of Mathieu functions. His method is that of separation of variables, after writing the heat equation in terms of elliptic coordinates, followed by the expansion of the constant θ_0 in a series of products of Mathieu functions. The same boundary value problem is also interpreted as one in the magnetic force inside a long solid elliptic core about which a solenoid of wire is wound, when a steady current is suddenly set up in the wire.

R. V. Churchill (Ann Arbor, Mich.).

Cherpakov, P. V. On heat delivery by a cylinder in a potential flow. C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 399–400 (1946).

Jaeger, J. C. Some applications of the repeated integrals of the error function. Quart. Appl. Math. 4, 100–103 (1946). [MF 15946]

The inverse Laplace transform of the function $s^{-n/2}e^{-s^{1/2}}(s^{1/2}+h)^{-1}$ is expressed in terms of iterated integrals of the complementary error function, integrals whose values were tabulated by D. R. Hartree [Proc. Manchester Lit. Philos. Soc. 80, 85–102 (1936)]. The author writes the solutions of a number of heat conduction problems of practical interest in terms of these integrals.

R. V. Churchill (Ann Arbor, Mich.).

Mitchell, Josephine. Some properties of solutions of partial differential equations given by their series development. Duke Math. J. 13, 87–104 (1946). [MF 15878]

The author considers the second order equation

$$\text{(I)}\qquad L_1[U(z, z^*)]=4U_{zz^*}+2aU_z+2a^*U_{z^*}+CU=0,$$

where subscripts denote partial differentiation, $z=x+iy$, $z^*=x-iy$ and a and c are entire functions of x and y, a^* being the conjugate of a; and the fourth order equation

$$\text{(II)}\qquad L_2[U(z, z^*)]=U_{zzz^*z^*}+MU_{zz}+RU_{zz^*}+M^*U_{z^*z^*}+AU_z+A^*U_{z^*}+CU=0,$$

where the coefficients are entire functions of x and y, and R and C are real. In a series of papers [notably, Trans. Amer. Math. Soc. 53, 130–155 (1943); 57, 299–331 (1945); Duke Math. J. 11, 617–649 (1944); these Rev. 4, 159; 7, 16; 6, 69] Bergman introduced classes of functions yielding the totality of solutions of (I) and (II) and obtained some of their function-theory properties. By this approach the author obtains sufficiency conditions for the continuity of solutions on simple closed curves in terms of the subsequences $\{D_{m0}\}$ and $\{a_{m0}\}$, where $U(z, z^*)=\sum_{m,n=0}^{\infty}D_{mn}z^m z^{*n}$ and $a=\sum_{mn=0}^{\infty}z^m z^{*n}$. Sufficiency conditions are also given for U to have a jump on the boundary and the size of the jump is given in terms of $\{D_{m0}\}$ and $\{a_{m0}\}$. Furthermore, summability conditions for the solution are given by the following theorem. If the series $\sum_{m=0}^{\infty}(2D_{m0}-\alpha_{m0}D_{00})$ is summable (C, α), $\alpha>-1$, and the series $\sum_{m=0}^{\infty}|2D_{m0}-\alpha_{m0}D_{00}|^2$ converges, then $\lim_{r\to1}\sum_{m,n=0}^{\infty}D_{mn}z^m z^{*n}=U(z, z^*)$, $|z|=1$, where $r\to1$ along any path lying between two chords of the unit circle $|z|=1$ which pass through the point $z=1$.

A. Gelbart (Syracuse, N. Y.).

Mambriani, Antonio. La derivazione parziale d'ordine qualunque e la risoluzione dell'equazione di Euler e Poisson. Ann. Scuola Norm. Super. Pisa (2) 11, 79–97 (1942). [MF 16754]

The author generalizes the classical partial differential equation of Euler [Institutionum Calculi Integralis, v. 3, St. Petersburg, 1770, pp. 262–271] and Poisson [J. École Polytech. 12, 215–248 (1823)], namely

$$\text{(E)} \qquad \frac{\partial^2 z}{\partial x \partial y}+\frac{\alpha}{x-y}\frac{\partial z}{\partial y}-\frac{\beta}{x-y}\frac{\partial z}{\partial x}=0,$$

where α, β are constants, through the use of the Liouville-Riemann differential operator of arbitrary order,

$$D^{\omega}_{x_0}f(x)=(1/\Gamma(-\omega))\int_{x_0}^{x}(x-t)^{-\omega-1}f(t)dt,$$

ω real or complex. [Reference is made to a previous paper by the author in Boll. Un. Mat. Ital. (2) 3, 9–18 (1940) [these Rev. 3, 119], where a similar application of the generalized derivative is made to the hypergeometric differential equation.]

Use is made of the fact that equation (E) remains invariant under the substitutions S_1 and S_2, where

$$S_1=\begin{pmatrix}\alpha, & \beta, & x, & y\\ \beta, & \alpha, & y, & x\end{pmatrix}, \quad S_2=\begin{pmatrix}\alpha, & \beta, & z\\ 1-\beta, & 1-\alpha, & (x-y)^{\alpha+\beta-1}z\end{pmatrix},$$

to show that (E) may be generalized in six different ways:

$$\begin{array}{ll}
(1) & D_x^{\alpha}D_y^{\beta}(x-y)D_y^{1-\beta}D_x^{1-\alpha}z=0,\\
(2) & D_x^{\alpha}(x-y)^{1-\beta}D_y(x-y)^{\beta}D_x^{1-\alpha}z=0,\\
(3) & D_y^{\beta}(x-y)^{1-\alpha}D_x(x-y)^{\alpha}D_y^{1-\beta}z=0,\\
(4) & D_x^{1-\beta}D_y^{1-\alpha}(x-y)D_y^{\alpha}D_x^{\beta}(x-y)^{\alpha+\beta-1}z=0,\\
(5) & D_x^{1-\beta}(x-y)^{\alpha}D_y(x-y)^{1-\alpha}D_x^{\beta}(x-y)^{\alpha+\beta-1}z=0,\\
(6) & D_y^{1-\alpha}(x-y)^{\beta}D_x(x-y)^{1-\beta}D_y^{\alpha}(x-y)^{\alpha+\beta-1}z=0.
\end{array}$$

Thus (1) coincides with (E) in case α, β are positive integers, while (4) coincides with (E) in case α, β are integers not exceeding 1.

The solutions of equations (1)–(6) are discussed and it is shown that they include particular solutions due to Darboux [Leçons sur la Théorie Génerale des Surfaces, 2d ed., v. 2, Paris, 1915, pp. 54–70], Appell [Bull. Sci. Math. Astron. (2) 6, 314–318 (1882)] and Buhl [J. Math. Pures Appl. (5) 10, 85–129 (1904)]. Thus, from the author's solution of (1) it is shown that, if α, β are positive integers, a solution of (E) is given by

$$z(x, y)=\frac{\partial^{\alpha+\beta-2}}{\partial x^{\alpha-1}\partial y^{\beta-1}}\left(\frac{X-Y}{x-y}\right),$$

where $X=X(x)$, $Y=Y(y)$ are arbitrary functions of the indicated variables. This is a result obtained by Darboux [loc. cit., p. 65]. Other similar results are noted and some illustrative examples are given. *M. A. Basoco.*

GEOMETRY

Busemann, Herbert. Metrically homogeneous spaces. Amer. J. Math. 68, 340–344 (1946). [MF 16431]

Cette note est consacrée au théorème suivant. Une condition nécessaire et suffisante pour qu'un espace distancié R soit localement isométrique à un espace euclidien, hyperbolique ou sphérique à un nombre fini de dimensions est que R soit compact à distance finie, convexe, localement extérieurement convexe et satisfasse localement à une relation de Stewart, entendant par là que $(\mathfrak{S})$ en tout pointe de R il existe un $\rho_a>0$ et une fonction $\Phi_a(\xi_1, \xi_2, \xi_3, \xi_4)$ telle que pour quatre points quelconques p, q, r, x de la sphère ouverte $S(a, \rho_a)$ avec $qx+\epsilon\cdot xr=qr>0$, $\epsilon=\pm 1$, nous ayons $px=\Phi_a(pq, pr, xq, \epsilon\cdot xr)$. L'auteur prouve aisément que sous les hypothèses énoncées, le prolongement en petit des segments est unique et par suite que l'espace R est un espace de type G [Busemann, Trans. Amer. Math. Soc. 56, 200–274 (1944), particulièrement p. 227; ces Rev. 6, 97]. Si p et p' désignent deux points distincts de $S(a, \rho_a/2)$, $B(p, p')$ la médiatrice de pp' (lieu des points y tels que $py=p'y$), q et r deux points de l'intersection $S(a, \rho_a/2) \cap B(p, p')$, $\mathfrak{s}=\mathfrak{s}(q, r)$ un segment joignant q et r, x un point quelconque de $\mathfrak{s}$, alors $\mathfrak{s}$ est inclus dans $S(a, \rho_a)$ et à cause de $(\mathfrak{S})$,

$$px=\Phi_a(pq, pr, xq, xr)=\Phi_a(p'q, p'r, xq, xr)=p'x$$

de sorte que $\mathfrak{s}(q, r) \subset B(p, p')$. La condition de "linéarité" de la médiatrice [loc. cit. p. 262] est vérifiée, d'après un résultat antérieur de l'auteur [loc. cit. p. 268], R est bien localement isométrique à un espace euclidien, hyperbolique ou sphérique de dimension finie.

Le théorème cesse d'être valable si l'on suppose $(\mathfrak{S})$ seulement pour $\epsilon=1$ ainsi que le montre l'exemple de l'espace constitué par les trois rayons $0\leqq\tau^i<\infty$, $i=1, 2, 3$, à origines identifiées, métrisé par $\tau_1^i\tau_2^j=|\tau_1^i-\tau_2^j|$, $i=j$; $\tau_1^i\tau_2^j=\tau_1^i+\tau_2^j$, $i\neq j$. Ici $px=\max(pq-xq, pr-xr)$ si $qx+xr=qr$. *C. Pauc* (Marseille).

Blumenthal, Leonard M. Metric characterization of elliptic space. Trans. Amer. Math. Soc. 59, 381–400 (1946). [MF 16462]

Les espaces euclidiens, hyperboliques et sphériques ainsi que leurs sous-ensembles ont été caractérisés métriquement par Menger et l'auteur [cf. Blumenthal, Distance Geometries, University of Missouri Studies, vol. 13 (1938)]. Dans cet article sont résolus les mêmes problèmes pour les espaces elliptiques $E_{n,r}$ (n: dimension éventuellement infinie, $\pi r/2$: diamètre). L'introduction de méthodes nouvelles est due à l'apparition de phénomènes géométriques que ne présentaient pas les espaces précités. Une congruence entre deux sous-ensembles de $E_{n,r}$ n'est pas nécessairement prolongeable en une congruence de l'espace sur lui-même. La dimension linéaire d'un ensemble de $E_{n,r}$ n'est pas déterminée univoquement par les distances mutuelles de ses points (exemple: triplet p, q, s avec $pq=qs=sp=\pi r/3$). Le résultat suivant permet de comprendre la nature des modifications apportées aux conditions concernant les espaces sphériques. Une condition nécessaire et suffisante pour qu'un m-tuple semimétrique $p_1, \cdots, p_m$ soit métriquement plongeable dans l'espace elliptique $E_{n,r}$ est que $0\leqq p_ip_j\leqq\pi r/2$, $i, j=1, 2, \cdots, m$, et qu'il existe une matrice carrée symétrique $\epsilon=(\epsilon_{ij})$, $\epsilon_{ij}=\epsilon_{ji}=\pm 1$, $\epsilon_{ii}=1$, telle que le déterminant $|\epsilon_{ij}\cos(p_ip_j/r)|$ ait un rang non supérieur à $n+1$ et que tous ses mineurs principaux non nuls soient positifs. Les espaces elliptiques $E_{n,r}$ pour r fixé sont caractérisés comme des espaces semimétriques de diamètre non supérieur à $\pi r/2$, convexes, complets, satisfaisant à une condition de convexité externe faisant intervenir des points antipodes et à des inégalités portant sur les mineurs des déterminants $|\epsilon_{ij}\cos(p_ip_j/r)|$, $i, j=0, 1, \cdots, 4$, les points $p_0, \cdots, p_4$ formant un quintuple admettant deux triplets linéaires. La caractérisation de $E_{n,r}$ pour n fini et fixé est obtenue par l'adjonction d'une condition d'annulation de determinants $|\epsilon_{ij}\cos(p_ip_j/r)|$, $i, j=0, 1, \cdots, k+1$, $k\geqq n$. *C. Pauc* (Marseille).

Hjelmslev, Johannes. Einleitung in die allgemeine Kongruenzlehre. V. Danske Vid. Selsk. Mat.-Fys. Medd. 22, no. 13, 37 pp. (1945).

This is a continuation of the third paper in this series [same Medd. 19, no. 12 (1942); these Rev. 7, 472]. It deals with a plane geometry in which two points can always be connected by a straight line but the intersection of a set of two or more different straight lines may be a whole segment. Two points that lie on such a "singular" segment were called neighbors. Two straight lines were called neighbors if each point of one of them has a neighbor point on the other. Neighborhood is transitive. Congruence was defined by means of a group of incidence-preserving transformations, transitive with respect to both the points and the straight lines. Special involutionary transformations served to define orthogonality.

A "weak" transformation maps each point either onto itself or a neighbor point. A transformation is weak if and only if it is direct and if there are two nonneighbor points that are neighbors of their image points. Each fixed point (or straight line) has a whole set of neighbor fixed points (or straight lines). The weak transformations without fixed points are studied in detail.

Given a singular segment S containing the points A and B, by rotating S around A we obtain a set $F=F(S)$ of neighbor points that is independent of the choice of A. It can be characterized as the set of those points that are fixed points of all the transformations that leave A and B fixed. We construct the normal of a straight line through S at a nonneighbor point of A on that line. The points of intersection of this normal with all the straight lines through S form a singular segment $\Sigma(S)$ which is determined by S alone up to congruences. Then S and $\Sigma(S)$ are reciprocal in the following sense: if S' is congruent to $\Sigma(S)$, then $\Sigma(S')$ is congruent to S. A similar theorem holds for the sets F. The intersection of a circle of nonsingular radius with one of its tangents is a singular segment Σ. Obviously, the intersection of the radii to the points of Σ is a singular segment S reciprocal to Σ. In addition, every subsegment of S is congruent to a suitable subsegment of Σ.

One paragraph deals with nonintersecting neighbor straight lines. Two results may be mentioned. Given two such lines p and p', if a straight line has exactly one point in common with each of p and p', then these two points are the end points of a subsegment of a singular segment. Up to congruences, this subsegment is determined by p and p' alone. The straight lines p and p' have either infinitely many or no common perpendiculars.

If no two straight lines have identical sets of normals, then (1) two adjacent sides of a rectangle are always reciprocal; (2) any two reciprocal segments may be the adjacent sides of a rectangle. If this condition is not satisfied, then either (a) any two straight lines with one common normal have the same normals or at least (b) each straight line has a set of neighbors with the same normals. In case (a) the sum of the angles of any triangle is equal to two right angles and any two segments can be the adjacent sides of a rectangle. The author extends Hilbert's "Streckenrechnung" to this geometry. "Es geht hieraus hervor, dass die Hauptsätze der Euklidischen Geometrie tatsächlich von dem Eindeutigkeitsaxiom unabhängig bestehen." In case (b), the sum of the angles of a triangle differs from two right angles only by a "singular" angle, that is, an angle whose sides lie on neighbor straight lines. Then (2) but not (1) remains valid. A weaker condition is substituted that is not only sufficient but also necessary. *P. Scherk.*

Hjelmslev, Johannes. A lecture on non-Euclidean geometry. Mat. Tidsskr. A. 1945, 3–36 (1945). (Danish)

Manara, Carlo Felice. Il parallelismo di Levi-Civita nel piano iperbolico. Period. Mat. (4) 23, 73–84 (1943).

Emch, Arnold. Endlichgleiche Zerschneidung von Parallelotopen in gewöhnlichen und höhern Euklidischen Räumen. Comment. Math. Helv. 18, 224–231 (1946). [MF 16412]

Extension to n dimensions of the equivalence by addition of two parallelograms of equal area or of two parallelepipeds of equal volume. The author states that the equivalence by addition of two polygons of equal area was proved in 1807 by William Wallace [Leybourn's Mathematical Repository, N.S. 3, 44–46 (1814): question 269 by Wallace, solution by Lowry]. *B. Jessen* (Copenhagen).

Tietze, Heinrich. Rekursionsformeln für den Inhalt gewisser Polyeder. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1941, 193–200 (1941).

In Euclidean n-space, take l points P_λ, $\lambda=1,\cdots,l$, whose rectangular Cartesian coordinates $(x_1^\lambda,\cdots,x_n^\lambda)$ are all positive. Let $k(P_\lambda)$ denote the orthotope (or rectangular parallelotope) which has OP_λ for a diagonal and edges parallel to the axes. Let $V_n(P_1,\cdots,P_l)$ denote the content of the sum of the l orthotopes $k(P_\lambda)$, with overlapping parts counted only once; for example, if P_1 is (1, 2) and P_2 is (2, 1), $V_2(P_1,P_2)=3$. Let the l points be so numbered that $x_n^\lambda\geqq x_n^{\lambda+1}$, $\lambda=1,\cdots,l$, with the convention that $x_n^{l+1}=0$, and let P_λ^* denote the projection of P_λ on the hyperplane $x_n=0$. The computation is gradually reduced to the extreme cases $V_n(P_1)=x_1^1\cdots x_n^1$ and $V_1(P_1,\cdots,P_l)=x_1^1$ by means of the recursion formula

$$V_n(P_1,\cdots,P_l)=\sum_{\lambda=1}^{l}(x_n^\lambda-x_n^{\lambda+1})V_{n-1}(P_1^*,\cdots,P_\lambda^*).$$

H. S. M. Coxeter (Notre Dame, Ind.).

Jensen, Henry. The twelve regular polyhedra. Mat. Tidsskr. A. 1945, 72–77 (1945). (Danish)

A polyhedron is said to be regular if its faces are regular polygons and its corners are regular solid angles. If we include as possible faces the polygons of higher density, such as the star pentagon or pentagram, and if we allow the faces to penetrate one another, the above definition admits not only the five Platonic solids but also the four star polyhedra of Kepler and Poinsot. Each of these has the same vertices as a Platonic solid, its "case," and has the same face-planes as a Platonic solid, its "core." By allowing the polyhedron to be discontinuous, while still insisting on a Platonic case and core, the author obtains three compounds: Kepler's "stella octangula" consisting of two interpenetrating tetrahedra, and a pair of enantiomorphous figures each consisting of five interpenetrating tetrahedra [Hess, Schriften der Gesellschaft zur Beförderung der Gesammten Naturwissenschaften zu Marburg 11, no. 1 (1876), p. 45]. He provides excellent drawings of these compounds and instructions for making paper models.

H. S. M. Coxeter (Notre Dame, Ind.).

Bacchiani, R. Sulla geometria del triangolo. Period. Mat. (4) 23, 37–39 (1943).

Cavallaro, Vincenzo G. Su le direttrici dell'ellisse di Brocard. Period. Mat. (4) 23, 40–48 (1943).

Thébault, V. Sur un théorème de Malet. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 84–86 (1946).

La présente note a pour but de généraliser le théorème suivant de Malet. Si des parallèles menées par les sommets d'un tétraèdre $ABCD$ rencontrent les faces opposées aux points A_1, B_1, C_1, D_1, le volume du tétraèdre $A_1B_1C_1D_1$ est, au signe près, triple de celui du tétraèdre $ABCD$.

From the author's summary.

Springer, C. E. Volume coordinates. Amer. Math. Monthly 53, 377–383 (1946).

Ghircoiaşiu, Nicolae. Une équation fonctionnelle caractérisant les coniques. Mathematica, Timişoara 22, 66–68 (1946).

Beginning with the well-known determinant formula for the equation of a conic through five points the author obtains a necessary and sufficient condition that the points (x, y_1), $(x+h, y_2)$, $\cdots$, $(x+4h, y_5)$ lie on a conic. The condition is the vanishing of a rather complicated polynomial in Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$, $\Delta^5 y$. All the work is elementary.

T. Fort (Athens, Ga.).

Conte, Luigi. Un problema relativo alla parabola secondo Fermat, Newton e Castillon. Period. Mat. (4) 22, 70–90 (1942).

The problem with which we are concerned is the following: to draw a parabola through four given points.

Extract from the paper.

Chariar, V. R. On the harmonic transversals of two pairs of quadrics and a straight line. J. Indian Math. Soc. (N.S.) 9, 46–50 (1945).

Mayer, O. Sur la composition des groupes projectifs et l'orientation des espaces projectifs réels. Bull. Math. Soc. Roumaine Sci. 46, 3–12 (1944). [MF 16507]

Toute théorie de l'orientation est liée à la structure du groupe de transformations qui définit la géométrie considérée. La théorie de la composition des collinéations est ici traité par les moyens élémentaires propres à la géométrie projective. Quand, dans une collinéation, on porte son attention sur l'un des points doubles (correspondant à l'une des valeurs spectrales de la matrice représentative), chacune des autres valeurs spectrales donne lieu, par rapport à la première, à un invariant absolu de la collinéation. Lorsque deux collinéations ont un même point double, leur produit admet également ce point double, avec un produit des invariants absolus égal au produit des produits analogues relatifs à chacune des collinéations composantes. Cela permet en particulier de décomposer en le produit de $r-m$ homologies paraboliques toute collinéation C de S_r possédant un S_m de points doubles, si le produit des invariants absolus de C est égal à 1, même si, pour C, tous les espaces linéaires S_{m+1} issus de S_m sont doubles.

Pour passer d'une collinéation quelconque (dont le produit des invariants absolus est égal à A) à une collinéation dont le produit des invariants soit 1, on peut utiliser une homologie parabolique auxiliaire, à condition de pouvoir extraire la racine d'ordre $r+1$ de A, ce qui est toujours possible dans le domaine complexe, ou dans le domaine réel pour r pair, mais dans le domaine réel pour r impair seulement si A est positif. Ainsi, toute collinéation dans un S_r peut être décomposée en un produit d'au plus $r+1$ homologies paraboliques, avec l'exception des collinéations indirectes dans les S_r réels de dimension impaire.

Il est ensuite montré que tout sous-groupe invariant propre d'un S_r réel ou complexe contient au moins une homologie parabolique, donc toutes les homologies paraboliques. Donc, le groupe des collinéations d'un S_r complexe, ou d'un S_r réel de dimension paire, est simple. De même, le groupe des collinéations directes d'un espace réel de dimension impaire est simple. *P. Belgodère* (Paris).

Turri, T. Sulle anticorrelazioni. Rend. Sem. Fac. Sci. Univ. Cagliari 10, 33–34 (1940). [MF 16215]

The author announces some of the results which he has proved in a forthcoming paper. These concern the composition of anticorrelations, the separation of irreducible anticorrelations into two general types and a determination of the number of projectively distinct anticorrelations whose squares are a given homography. *J. G. Semple.*

Turri, Tullio. Il numero delle schiere di proiettività e antiproiettività permutabili con un'omografia data. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 55–62 (1944). [MF 16633]

Les conditions pour qu'une homographie donnée soit permutable avec une antihomographie ou une anticorrélation (au sens de la géométrie projective complexe) sont déterminées. Les résultats obtenus sont appliqués au calcul du nombre de familles continues d'homographies, corrélations, antihomographies et anticorrélations qui composent le groupe Γ des transformations permutables avec une homographie donnée α. Si α n'est pas une homographie de Del Prete (racines caractéristiques formées, avec des diviseurs élémentaires de mêmes degrés, par des familles de nombres proportionnels aux puissances d'une même racine primitive, d'indice q, de l'unité), le nombre de familles continues qui composent le groupe Γ des projectivités et antiprojectivités permutables avec α peut avoir l'une des valeurs 1, 2 ou 4. Si α est une homographie de Del Prete, q étant l'ordre maximum des homographies cycliques qui sont permutables avec α et échangent ses espaces caractéristiques, ce nombre de familles continues peut avoir l'une des valeurs q, $2q$ ou $4q$.

P. Belgodère (Paris).

Turri, Tullio. Correlazioni proiettivamente distinte le cui omografie quadrato sono proiettivamente identiche. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 63–70 (1944). [MF 16634]

Une note précédente de l'auteur, de même titre [Rend. Circ. Mat. Palermo 58, 175–189 (1934)] est simplifiée et complétée. Les racines de l'équation caractéristique de α^2 se groupent en familles faisant intervenir les puissances paires et impaires d'une racine primitive d'indice $2q$ de l'unité. Si α et β sont deux corrélations projectivement différentes et telles que α^2 et β^2 soient deux homographies projectivement égales, toute autre corrélation γ dont le carré γ^2 est projectivement égal à α^2 et β^2 est projectivement égale à α ou à β. L'homographie α^2 est une homographie particulière de Del Prete (permutable avec au moins une homographie cyclique d'ordre q qui échange entre eux les espaces caractéristiques de α^2). Elle est donc permutable avec q familles continues d'homographies et avec q familles continues de corrélations, dont l'une contient α. Si, de plus, α est réelle (ou, ce qui revient au même, de type réel), α^2 est également permutable avec q familles continues d'antihomographies et q familles continues d'anticorrélations.

P. Belgodère (Paris).

Turri, Tullio. Sulle antiinvoluzioni di prima specie trasformanti in sè omografie e correlazioni reali. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 71–79 (1944). [MF 16635]

Étant donnée une projectivité réelle α (homographie ou corrélation), il peut exister des antiinvolutions de première espèce (distinctes de la conjugaison par rapport à l'espace ambiant et de ses transformées par les homographies permutables avec α) transformant α en elle même. Cela revient à rechercher les couples de projectivités α et β, projectivement égales dans le champ complexe et projectivement différentes dans le champ réel. Les résultats de l'auteur [Rend. Sem. Mat. Univ. Padova 7, 111–127 (1936); Ann. Mat. Pura Appl. (4) 13, 143–161 (1934)] sont repris et complétés, pour obtenir les conditions d'existence et la construction effective des antiinvolutions recherchées, selon les types des projectivités considérées. *P. Belgodère* (Paris).

Turri, Tullio. La continuità delle trasformazioni conservanti i gruppi armonici sulla retta proiettiva complessa e la continuità degli automorfismi del gruppo delle omografie in un S_n complesso. Rend. Sem. Fac. Sci. Univ. Cagliari 15, 2–15 (1946). [MF 16395]

L'auteur précise le rôle de la continuité dans certaines questions étudiées par É. Cartan [Leçons sur la Géométrie Projective Complexe, Gauthier-Villars, Paris, 1931]. É. Cartan montre que les automorphies du groupe des homographies et des antihomographies de la droite projective complexe sont continues. L'auteur du présent travail remarque qu'il suffit, avec É. Cartan, d'observer qu'une automorphie du groupe précédent transforme une homographie en une homographie et une antihomographie en une antihomographie, pour déduire du résultat ci-dessus la continuité des automorphies du groupe formé par les seules homographies de la droite projective complexe. Puis il complète un résultat relatif aux transformations de la droite projective complexe conservant les groupes harmoniques de quatre points, en montrant, qu'étant donnée une transformation ponctuelle univoque de la droite transformant deux points distincts en deux points distincts et un groupe harmonique en un groupe harmonique, il n'est pas nécessaire de supposer la continuité de la transformation pour pouvoir affirmer que cette transformation est une homographie ou une antihomographie.

L'auteur étend ensuite à un espace projectif complexe S_n le théorème fondamental de la géométrie projective dans S_1, donnant les conditions pour qu'une transformation ponctuelle de cet espace soit une homographie ou une antihomographie. Il montre que l'hypothèse de la continuité peut être omise, et, calquant le raisonnement de É. Cartan dans S_1, il obtient la proposition générale suivante: si une transformation ponctuelle T d'un espace projectif complexe S_n $(n>1)$ jouit des propriétés, d'être univoque, de transformer deux points distincts en deux points distincts, $n+1$ points non situés dans un même hyperplan en $n+1$ points non situés dans un même hyperplan, et $n+1$ points d'un même hyperplan en $n+1$ points d'un même hyperplan, T est une homographie ou une antihomographie.

Enfin, s'inspirant toujours des raisonnements de É. Cartan dans S_1, mettant en jeu les propriétés des involutions centrales, l'auteur établit que, dans S_n, les automorphies du groupe des homographies proviennent d'une projectivité ou d'une antiprojectivité, d'où il résulte que les automorphies du groupe des homographies de S_n sont continues.

P. Vincensini (Besançon).

Belgodère, Paul. Correspondance involutive sur une conique. Généralisation. Euclides, Madrid 6, no. 59, 24–29 (1946).

Examen de différentes extensions, métriques ou projectives, aux coniques quadriques et cubiques gauches, du théorème: deux points en involution sur une conique sont, de ce fait, conjugués par rapport à un cercle fixe.

Author's summary.

Rindi, Scipione. Sui punti uniti nelle forme proiettive sovrapposte concordi. Period. Mat. (4) 23, 34–36 (1943).

Giuseppina, Casara. Delle coniche come luoghi geometrici relativi ad alcuni problemi di contatto. Period. Mat. (4) 22, 173–177 (1942).

Fabricius-Bjerre, Fr. The theory of conic sections on the sphere. Mat. Tidsskr. A. 1945, 53–71 (1945). (Danish)

An elementary and unified derivation of the (well-known) properties of spherical conics. *H. Busemann.*

Tibiletti, Cesarina. Sul problema di Apollonio: una soluzione spaziale. Period. Mat. (4) 24, 20–39 (1946).

Ghurye, S. G. The conical projection of a circle. J. Univ. Bombay (N.S.) 14, part 5, 1–5 (1946).

Mosharrafa, A. M. Conical transformations. Proc. Math. Phys. Soc. Egypt 2, 21–28 (1944). [MF 16832]

The author summarizes his paper as follows. If P and P' are two points on the same generator of a given cone, the transformation from the coördinates (x_1, x_2) of the point P referred to axes in a given plane passing through P to the coördinates (x_1', x_2') of the point P' referred to axes in a corresponding plane passing through P' is called a conical transformation. The analysis gives explicit expressions for the transformation from (x_1, x_2) to (x_1', x_2'). The work is generalized to n dimensions. It is shown that conical transformations form a continuous one-parameter group.

C. C. MacDuffee (Río Piedras, P. R.).

Buttgenbach, H. Problèmes de projections stéréographiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 455–461 (1946).

The author derives a graphical method for finding the stereographic projection of a circle. *E. Lukacs.*

Campedelli, Luigi. Una visione sintetica proiettiva dei metodi descrittivi. Period. Mat. (4) 24, 10–19 (1946).

de Rafael, Enrique. Exact axonometric scales. Publ. Inst. Mat. Univ. Nac. Litoral 6, 145–167 (1946). (Spanish) [MF 16918]

In the axonometric projection used in descriptive geometry three mutually perpendicular axes are projected orthogonally upon a plane. If the reductions in scale from each of the axes on the picture plane are m, n, p, respectively, it is known that (1) $m^2+n^2+p^2=2$, where in addition m, n, p are positive and less than unity. This paper deals with projections where the scales are rational ("exact"), that is, essentially, with rational solutions of (1) or integral solutions of the corresponding equation $m^2+n^2+p^2=2q^2$. Several special cases are discussed (for example, $m=n$); some of these lead to a Pell equation. [The author does not give the general solution of (1) in terms of rational parameters, although this would have at once made obvious the correctness of his

conjecture that, given any axonometric projection (with rational or irrational scales), it is possible to find a projection with rational scales that differ from those of the given projection by less than an arbitrary amount.]

H. W. Brinkmann (Swarthmore, Pa.).

Finsterwalder, Sebastian. Der Folgebildanschluss. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 91–110 (1941).

The author considers the problem of photogrammetric reconstruction under the restriction that two points are fixed with respect to one center of projection. This problem occurs in aerial surveying when a new photograph has to be fitted to a picture which was taken previously and covered part of the same region.

This paper was discussed by W. Wunderlich [Monatsh. Math. Phys. **51**, 57–58 (1943); these Rev. **7**, 259].

E. Lukacs (Cincinnati, Ohio).

Yarden, Dov. Classes of areas of spherical triangles which correspond to the trihedral angles of a tetrahedron. Riveon Lematematika **1**, 7 (1946). (Hebrew)

Tuchman, Zevulon. A simple proof of the theorem on the sum of the measures of the external trihedral angles of a tetrahedron. Riveon Lematematika **1**, 20 (1946). (Hebrew)

Algebraic Geometry

van der Waerden, B. L. The foundation of the invariant theory of linear systems of curves on an algebraic surface. Nederl. Akad. Wetensch., Proc. **49**, 223–226 = Indagationes Math. **8**, 120–123 (1946). [MF 16576]

One of the preliminary difficulties which Italian geometers had to overcome, before they could attempt to give a general theory of linear systems of curves on surfaces, was the introduction of base-points (with their "virtual" and "effective" multiplicities [cf., for example, O. Zariski, Algebraic Surfaces, Ergebnisse der Math., v. 3, no. 5, Springer, Berlin, 1935, chap. 2, §§ 1–4]). Zariski's well-known paper [Amer. J. Math. **60**, 151–204 (1938)] clearly indicated that conditions imposed by base-points would have to be replaced, in any generalization to higher dimensions, by conditions imposed by valuations. A method of formulating such conditions is proposed by the author, who states that his definition ensures the birational invariance of complete systems and of the sum and difference of such systems. It is not clear whether the existence of nonsingular models is assumed.

A. Weil (São Paulo).

Scott, D. B. Point-curve correspondences. II. Induced and extended correspondences. Proc. Cambridge Philos. Soc. **42**, 229–239 (1946).

This paper is a continuation of a preceding one [same Proc. **41**, 135–145 (1945); these Rev. **7**, 27]. Given a point-curve correspondence T between two algebraic surfaces F, G, represented by an algebraic threefold W on the product variety $F\times G$, there is an induced correspondence T^* between given curves C^*, D^* lying, respectively, on F, G which is represented by the intersection W^* of $C^*\times D^*$ and W. It is shown that if W^* is irreducible and not degenerate then every 1-cycle of C^* is transformed into an invariant 1-cycle on D^*. The author next considers the question: given curves C^* and D^* on F and G, and an irreducible correspondence S^* between C^* and D^*, does there exist a point-curve correspondence T between F and G such that the induced correspondence T^* is homologous to S^* as a correspondence between C^* and D^*? If such a correspondence T exists it is called an extension of S^*. A fractional extension T of S^* is a correspondence such that T^* plus a suitable correspondence of valency zero is homologous to a multiple of S^*. It is shown that, if C^* and D^* are proper curves, a correspondence S^* has at least a fractional extension if, and only if, it is an invariant correspondence, that is, one in which the transformation matrix of the cycles of C^* and D^* involves only the invariant cycles. It is pointed out that for sufficiently general curves C^* and D^* the only correspondences between them are invariant. The author next considers point-curve correspondences between a curve and a surface and shows how to obtain a base for them. The paper ends with a generalisation of the idea of an extended correspondence.

J. A. Todd.

Ancochea, Germán. Courbes algébriques sur corps fermés de caractéristique quelconque. Acta Salmanticensia **1**, 39 pp. (1946). [MF 16695]

One of the methods of developing the theory of algebraic curves over an algebraically closed field K of characteristic zero is by the use of formal power series over K. The basic theorem in this development is the following. If $F(x, z)$ is a polynomial in z whose coefficients are power series in x then, for some integer n, $F(t^n, z)$ can be factored into linear factors of the form $z-f_i(t)$, where $f_i(t)$ are power series in t. In extending this method of development to fields K of arbitrary characteristic the author has given a proof of this theorem with the one change that t_n is replaced by a more general polynomial $f(t)$. The proof follows the one given by Ostrowski [Math. Z. **37**, 98–133 (1933)] for the case of characteristic zero. Most of the theory of curves can be developed from this theorem in the classical manner. One exception is the analysis of neighboring singularities based on the power series expansion. The usual method uses the relation $x=t^n$ and the passage to the more general $x=f(t)$ would introduce excessive complications. These are avoided by the use of an analysis based on an algorithm due to M. Noether [Rend. Circ. Mat. Palermo **4**, 89–108 (1890)].

R. J. Walker (Ithaca, N. Y.).

Châtelet, François. Introduction géométrique à l'étude arithmétique des cubiques planes. Revue Sci. **84**, 3–6 (1946).

The author states and partly proves various well-known elementary results pertaining to the geometry on a cubic curve of genus 1 (over an algebraically closed field of constants).

A. Weil (São Paulo).

Burau, Werner. Über zweifach unendliche rationale Mannigfaltigkeiten linearer Räume. Abh. Math. Sem. Hansischen Univ. **15**, 1–26 (1943). [MF 15826]

If the elements of a rational two-parameter family F_2 of linear spaces are mapped on a plane the lines of the plane correspond to certain rational one-parameter subfamilies F_1 of the set of spaces. The author investigates such F_2's by studying the structure of the F_1's contained in them. If the F_1's are normal varieties or cones over these it is shown in the simplest cases that the F_2's are either of a certain normal type or are obtainable from such a type by projection or section. The author states that the same methods can be used to obtain this result also for the more complicated cases.

R. J. Walker (Ithaca, N. Y.).

Bompiani, Enrico. Approssimazione di una superficie algebrica nell'intorno di una sua retta. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 68–75 (1943). [MF 16242]

The author shows that an algebraic surface F^n of order n in S_3 which has a single simple line r can always be approximated in the neighborhood of the first order of r by means of a ruled surface of the same order n which has r as a directrix and also an arbitrary $(n-1)$-fold rectilinear directrix. Results are also obtained when F^n has multiple points on r. The asymptotic tangents of F^n at points of r generate a ruled surface R^{3n-4} of order $3n-4$; F^n and R^{3n-4} have contact of the second order at every point of r.
T. R. Hollcroft (Aurora, N. Y.).

Rollero, Aldo. Su alcune rigate tangenti od osculatrici ad una superficie algebrica lungo una sua retta multipla. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 154–168 (1943). [MF 16244]

It is shown that an algebraic surface F^n of order n with a double line a can be approximated in the first order neighborhood of a by means of a ruled surface of order n and in the second order neighborhood of a by means of a ruled surface generated by lines which have four-point intersection with F^n at points of a. Modifications of these results are obtained when F^n has triple points on a. The cases when F^n has a cuspidal line and finally an r-fold line are treated. This is an extension to multiple lines of the results obtained by Bompiani [see the preceding review] for a simple line of F^n.
T. R. Hollcroft (Aurora, N. Y.).

Rollero, Aldo. Su alcune rigate tangenti ad una superficie algebrica lungo una sua conica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 79, 308–325 (1944). [MF 16235]

Two ruled surfaces Γ and $\bar{\Gamma}$ associated with an algebraic surface F^n of order n and containing a conic γ are studied. The surface Γ is the envelope of planes tangent to F^n at points of γ and is of order $4n-6$. The order of Γ is reduced when F^n has multiple points on γ. The other ruled surface $\bar{\Gamma}$ is of order $2n$ and has γ as a simple directrix and an arbitrary line as a $(2n-2)$-fold directrix. Then F^n can be approximated in a neighborhood of first order along γ by either Γ or $\bar{\Gamma}$. In this paper the methods and results of Bompiani and Rollero for lines of F^n [see the two preceding reviews] are extended to a conic. *T. R. Hollcroft.*

Morin, Ugo. Sui tipi di sistemi lineari di superficie algebriche a curva-caratteristica di genere due. Ann. Mat. Pura Appl. (4) 19, 257–288 (1940). [MF 15088]

Enriques has shown that a linear system of algebraic surfaces of S_3 with a variable characteristic curve of genus $\pi>1$ is birationally equivalent to a linear system of surfaces of order r with a basis line of multiplicity $r-2$, two basis points in the plane generated by this line and other basis elements. Using this result, the author treats the problem of obtaining all birationally distinct types of linear, complete, irreducible systems of algebraic surfaces of S_3 with a variable characteristic curve of genus $\pi=2$. He finds 25 such types or families of surfaces. The dimension, least order and equation of each family are given. *T. R. Hollcroft.*

Thalberg, Olaf M. Some remarkable theorems concerning intersections of algebraic curves. Avh. Norske Vid. Akad. Oslo. I. 1943, no. 4, 11 pp. (1943). [MF 16437]

The theorems include a far-reaching generalisation of the Bertini involution [cf. H. F. Baker, Principles of Geometry, vol. 6, Cambridge University Press, 1933, p. 124] and some examples of involutions in the plane in which each pair of corresponding points lies on a (variable) conic through four fixed points. *D. B. Scott* (London).

Burniat, Pol. Sur la réduction à l'ordre minimum des systèmes de courbes algébriques planes de genre quelconque. Ann. Sci. École Norm. Sup. (3) 62, 93–114 (1945).

The author considers linear systems of plane curves such that the genera of the general curve of the system and of the general curves of the successive pure adjoint systems form an arithmetic progression. He shows that a suitable birational transformation of the plane will reduce the curves of such a system to one or other of a specified number of types of lowest possible order. The actual results are too involved to quote in detail. *J. A. Todd.*

Pompilj, Giuseppe. Sulle varietà abeliane. Ann. Mat. Pura Appl. (4) 20, 271–289 (1941). [MF 16606]

Some known results on Abelian varieties, formerly obtained by transcendental means, are proved by algebraic means. *From the author's summary.*

Turri, T. Sui gruppi di moltiplicabilità delle varietà abeliane di tipo reale. Rend. Sem. Fac. Sci. Univ. Cagliari 11, 26–55 (1941). [MF 16218]

This is a detailed study of the multiplicity group G^* of an Abelian variety V_p. If S is the symmetry (involutory antibirational transformation) admitted by V_p, ω the Riemann matrix of V_p, τ the image of ω in Scorza's representation and $\bar{\tau}$ its conjugate, and if δ is the biaxial hyperbolic involution which represents the involutory unimodular substitution induced by S on the system of $2p$ primitive linear cycles on the Riemannian of V_p, then G^* corresponds to the group of rational homographies which leave τ and $\bar{\tau}$ fixed, while δ permutes τ, $\bar{\tau}$ and therefore transforms G^* into itself without belonging to it. The work falls into two parts: (i) the case in which ω has no isolated axes; (ii) the case in which ω has two isolated axes permuted by δ; furthermore, (i) subdivides according as the isolated spaces which are fixed for G^* are fixed for δ or interchanged in pairs by δ.

The paper discusses in some detail the question raised by Comessatti [Ann. Mat. Pura Appl. (4) 3, 27–71 (1925)] as to when V_p admits symmetries (with respect to birational transformations) other than products of S by ordinary transformations of the first and second species. In this case G^* must be transformed into itself by at least one hyperbolic biaxial unimodular involution μ, distinct from δ with respect to homographies of G^* with unimodular matrix, which is such that $\mu\delta$ is a collineation of G^*.
J. G. Semple (London).

Turri, Tullio. Sulla normalizzazione reale delle forme alternate principali di varietà abeliane reali. Rend. Sem. Fac. Sci. Univ. Cagliari 12, 21–33 (1942). [MF 16219]

If S is the substitution induced on a system of $2p$ primitive linear cycles of the Riemannian of a real Abelian variety whose principal alternating form is Φ, then the matrices of S and Φ can be reduced (by a unimodular substitution) to the forms

$$\left(\begin{array}{c|c} I & 0 \\ \hline C & -I \end{array}\right), \quad \left(\begin{array}{c|c} \frac{1}{2}(-AC+C'A') & A \\ \hline -A & 0 \end{array}\right),$$

respectively. Cherubino has shown [Atti Ist. Veneto Sci.

Lett. Arti 89, 271–289 (1930)] that a necessary condition for the real normalization of the matrix of Φ is that the elementary divisors of the latter should coincide with those of A. The author shows that for a given involutory unimodular linear substitution S and alternating form Φ with matrices of the above form there always exist corresponding real Riemann matrices; he then proves that Cherubino's condition is also sufficient for the real normalization of Φ. He also offers some criticism of Cherubino's work and points out an error in one of Cherubino's deductions from his theorem. *J. G. Semple* (London).

Cherubino, Salvatore. Sulla normalizzazione reale delle forme riemanniane principali. Rend. Sem. Fac. Sci. Univ. Cagliari 12, 69–74 (1942). [MF 16221]

The author replies in this paper to Turri's criticism [see the preceding review], maintaining that his calculations were rigorous, the only error being that pointed out by Turri in the actual enunciation of the deduction in question. He explains also that Turri's main result is contained, though only implicitly, in Cherubino's own development. *J. G. Semple* (London).

Cherubino, Salvatore. Un teorema sulle corrispondenze algebriche tra due curve. Ann. Scuola Norm. Super. Pisa (2) 11, 99–103 (1942). [MF 16755]

The properties of the algebraic correspondence between curves depend chiefly on the matrices T and T^* of the characteristic integers of the correspondence. In previous papers, the author has made a rigorous study of these matrices. In the present paper he finds a new property of the product TT^*, namely, that for two curves each of genus $p_i \geqq 1$, TT^* is always a matrix of canonical diagonal form. If two hyperelliptic curves are of the same genus, this result leads at once to a well-known condition due to Torelli for the birational equivalence of the two curves. *T. R. Hollcroft* (Aurora, N. Y.).

Sansone, G. Su un problema di analisi indeterminata e sui punti razionali di una famiglia di curve ellittiche dipendenti da un parametro. Ann. Mat. Pura Appl. (4) 20, 105–135 (1941). [MF 16598]

The following is an extract from the author's summary. The author proves that the family of elliptic cubics $C^3(v)$:

$$Y^2=4\left(X-\frac{v^4+1}{6}\right)\left(X+\frac{v^4-6v^2+1}{12}\right)\left(X+\frac{v^4+6v^2+1}{12}\right)$$

possesses, for every rational value of v, a configuration of eight rational points and shows that the existence of a ninth rational point on $C^3(v)$ implies the existence on that $C^3(v)$ of infinitely many rational points. He also shows that there exist rational values of v for which $C^3(v)$ possesses only eight rational points, and values of v, for example

$$v=(\xi-1)(\xi^2+3\xi+1)/(\xi+1)(\xi^2-3\xi+1)$$

with ξ rational, for which all $C^3(v)$ have infinitely many rational points. *D. B. Scott* (London).

Apéry, Roger. Sur certaines variétés algébriques à $(n-2)$ dimensions de l'espace à n dimensions. C. R. Acad. Sci. Paris 222, 778–780 (1946). [MF 16169]

A matrix variety V_{n-2} in $[n]$ is one which is given by the vanishing of all the h-rowed determinants of a matrix $\|A_{ij}\|$, $i=1, \cdots, h$; $j=1, \cdots, h+1$, the A_{ij} being homogeneous polynomials of suitably restricted degrees in the coordinates $x_0, \cdots, x_n$. The author shows that every V_{n-2} of $[n]$ which can be defined by a chain of intersections, starting from a complete intersection of primals, is a matrix V_{n-2}. *J. G. Semple* (London).

Ciani, E. Intorno alla quartica di Klein. Boll. Un. Mat. Ital. (2) 4, 129–133 (1942). [MF 16063]

L'auteur a étudié dans des travaux précédents certaines particularités de la quartique $xy^3+yz^3+zx^3=0$; il expose ici une suite de relations entre la courbe et ses formes covariantes et contravariantes. *B. Levi* (Rosario).

Fürle, František. Eine rationale Regelfläche sechsten Grades. Publ. Fac. Sci. Univ. Masaryk 1939, no. 274, 23 pp. (1939). (Czech. German summary)

The "hypopede" is defined as the quartic curve of intersection of a cylinder of radius a and a sphere of greater radius $a+b^2/a$ touching each other internally. Those bisecants of the hypopede which meet the axis of the cylinder are found to generate a ruled surface of order 6 and class 6, which has a double curve of order 10 consisting of the hypopede, a strophoid and three straight lines. The line of striction of the ruled surface is found to be a twisted octavic entirely contained within a cylinder of radius $a/4$. *H. S. M. Coxeter* (Notre Dame, Ind.).

Ciani, Edgardo. Intorno alla hessiana e cayleyana di una cubica piana. Period. Mat. (4) 24, 54–61 (1946).

Prior, L. E. The nodes of some cubic pencils. Amer. Math. Monthly 53, 366–370 (1946).

Derwidué, L. Sur les involutions de l'espace. Bull. Sci. Math. (2) 69, 129–137 (1945). [MF 15889]

Enriques a démontré par un exemple l'existence d'involutions non rationnelles de dimension supérieure à 2. L'auteur énonce un théorème qui donne un critère assez général de rationnalité. Si sur le courbe générique d'une congruence linéaire de l'espace à trois dimensions se trouve rationnellement déterminée une série lineaire d'ordre n et de dimension r, l'ensemble des groupes de ces séries forme une variété à $r+2$ dimensions. Il paraît que la démonstration devrait être précisée dans quelques points. *B. Levi* (Rosario).

Godeaux, Lucien. Sur une catégorie de surfaces algébrique doubles. Acta Acad. Ci. Lima 8, 139–153 (1945).

En se rattachant à ses recherches sur les surfaces algébriques qui sont l'image d'une involution du second ordre dépourvue de points unis appartenant à une surface algébrique [Ann. Fac. Sci. Univ. Toulouse (3) 5, 289–312 (1914); Les involutions cycliques appartenant à une surface algébrique, Actual. Sci. Ind., no. 270, Hermann, Paris, 1935], l'auteur considère ici, dans l'espace à trois dimensions, une surface F d'ordre $2n+1$ touchant un cône Q du second ordre le long d'une courbe D d'ordre $2n+1$. Une telle surface possède $4n^2$ points doubles coniques situés sur D; elle est l'image d'une involution I_2 d'ordre deux appartenant à une surface F^*; les $4n^2$ points doubles coniques de F sont les points de diramation de la correspondance entre F et F^*. En partant de la représentation du cône Q sur un plan (à l'aide du système linéaire de courbe du troisième ordre ayant un point double et passant par trois points d'une même droite), l'auteur établit l'équation de F, étudie les courbes tracées sur F et sur F^* et démontre que F et F^* sont deux surfaces régulieres. *E. G. Togliatti* (Gênes).

Godeaux, Lucien. Sur les involutions appartenant à des variétés algébriques intersections complètes d'hypersurfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 301–317 (1945).

Godeaux, Lucien. Sur les involutions cycliques régulières d'ordre trois appartenant à une surface irrégulière. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 134–147 (1946).

Godeaux, Lucien. Sur les involutions du septième ordre appartenant à une surface de genres un. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 148–163 (1946).

Godeaux, Lucien. Sur les courbes et surfaces intersections d'hyperquadriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 262–269 (1945).

Godeaux, Lucien. Sur les variétés algébriques à trois dimensions sur lesquelles l'opération d'adjonction a la période trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 318–326 (1945).

Godeaux, Lucien. Sur la construction d'une surface d'irrégularité deux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 11–18 (1945).

Godeaux, Lucien. Construction d'une surface algébrique d'irrégularité quatre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 9–16 (1946).

Godeaux, Lucien. Recherches sur la construction de surfaces algébriques irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 17–32 (1946).

Godeaux, Lucien. Construction d'une surface canonique du neuvième ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 202–212 (1945).

Godeaux, Lucien. Sur la construction des surfaces doubles n'ayant qu'un nombre fini de points de diramation. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 213–225 (1945).

Jongmans, F. Remarques sur la classification des variétés algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 414–429 (1946).

Nollet, Louis. Sur les surfaces algébriques possédant un faisceau linéaire de cubiques gauches. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 67–74 (1945).

Differential Geometry

✱Jessen, Børge. Lærebog i Geometri. II. Differentialgeometri og Nomografi. [Textbook of Geometry. II. Differential Geometry and Nomography]. Jul. Gjellerups Forlag, Copenhagen, 1941. viii+286 pp. (Danish)

✱Jessen, Børge. Lærebog i Geometri. II. Differentialgeometri og Nomografi. [Textbook of Geometry. II. Differential Geometry and Nomography]. 2d ed. Jul. Gjellerups Forlag, Copenhagen, 1945. viii+323 pp. (Danish)

The first three chapters (curves, motion of rigid figures, surfaces) serve as an introduction to elementary differential geometry. The more advanced methods of the Gauss or Darboux theories are not used and the second fundamental form is just mentioned. On the other hand, the material covered is treated with unusual rigor and thoroughness and at the same time in a most intuitive and natural way. Kinematical methods are in the foreground. However, there reigns a refreshing lack of methodological bias and the text keeps close to other subjects illustrating the problems from many angles. The last chapter contains an introduction to nomography and is independent of the first part. Its inclusion in the book is explained by the curricula for which the book is intended. An abundant number of examples accompany the text. In addition, there are 176 exercises, 158 of which refer to differential geometry. *W. Feller.*

Kasner, Edward, and De Cicco, John. The distortion of angles in general cartography. Proc. Nat. Acad. Sci. U. S. A. **32**, 94–97 (1946). [MF 16360]

Concerning the differential geometry of the map of a surface Σ on a plane Π, the authors discuss the azimuthal ratio $\alpha=d\Theta/d\theta$, or the rate of change of angle at a point on Σ with respect to the corresponding angle on Π; thus the azimuthal ratio is a function of the lineal element on Π. A curve along which the azimuthal ratio is constant is called an azimuthal curve. The scale $\sigma=ds/dS$ is the ratio of element of length on Π to corresponding element of length on Σ; a scale curve is a curve along which σ is constant. Azimuthal curves and scale curves are compared and contrasted. It is shown that $\alpha=H\sigma^2$, where $H^2=EG-F^2>0$. The totality of azimuthal curves and the totality of scale curves coincide if and only if the map is obtained by an equiareal mapping followed by a magnification; otherwise, the only curves which are simultaneously azimuthal and scale curves are the minimal lines on Π, the minimal curves on Σ and the curves $H=$constant. An arbitrary Σ can be mapped on Π in such a way that the azimuthal curves are straight lines on Π; but only for special surfaces can the scale curves be straight lines on Π. *E. F. Beckenbach.*

Löbell, Frank. Differentialinvarianten bei Flächenabbildungen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 217–237 (1944).

If a region without singular points on a surface S_1 is mapped by a regular transformation onto a similar region on a surface S_2 there is induced between the tangent planes at corresponding points a projective transformation or "affinity." The directions in which the mapping at a given point produces a maximum or minimum stretching are called principal directions and the ratios of stretching in these directions the principal ratios. After expressing the surfaces parametrically so that corresponding points have the same parameter values, the author establishes a set of vector and scalar invariants with respect to any regular change of parameters. He then shows that by these invariants the affinities between the tangent planes at corresponding points are completely established and that, in particular, both the principal ratios and the principal directions in space are determined by them. *S. B. Jackson.*

Hauer, F. Entwicklung der flächentreuen Abbildung kleiner Bereiche des Rotationsellipsoids in die Ebene bis einschliesslich Glieder 4. Ordnung. Z. Vermessungswesen **72**, 179–189 (1943).

Hristow, Wl. K. Transformationsformeln zwischen den Gauss-Krügerschen und den Soldnerschen Koordinaten. Z. Vermessungswesen 73, 157–165 (1944).

Mineo, Corradino. Sul passaggio da uno a un altro degli ellissoidi locali relativi a una data regione del geoide e sulle consequenti variazioni delle coordinate ellissoidiche dei vertici della rete geodetica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 333–337 (1941).

Gulotta, Beniamino. Sulla determinazione, per via algebrica, degli scostamenti lineari del geoide da un ellissoide locale. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 614–617 (1941).

Boaga, Giovanni. Sulla risoluzione dei triangoli geodetici ellissoidici attraverso la loro rappresentazione piana conforme. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 955–967 (1944).

Bouligand, Georges. Sur les liaisons isométriques. C. R. Acad. Sci. Paris 222, 1195–1197 (1946). [MF 16726]

From the introduction: "Dans l'espace euclidien à trois dimensions, soit une liaison L s'exerçant entre une courbe d'une première famille F_1 et une courbe d'une seconde famille F_2, et telle que, si ces courbes simultanément engendrent des surfaces, ces dernières se correspondent isométriquement, les courbes génératrices extraites de F_1 étant associées à celles extraites de F_2. On dit alors que L est isométrique. L'expression d'une telle liaison (non holonome) requiert des formes différentielles quadratiques." The paper contains some remarks on the case that both F_1 and F_2 are families of straight lines. *P. Scherk* (Saskatoon, Sask.).

Bouligand, Georges. Résolution opératoire de problèmes particuliers concernant les surfaces isométriques. Revue Sci. 83, 131–144 (1945). [MF 16983]

After some comments on the present status of the problem of finding all surfaces isometric to a given one, in particular, on the contributions of modern analysis, the author states the "problem *NG*," the problem of isometry in the case where the coordinates, as functions of parameters u and v, are degenerate (noyaux de Goursat), that is, finite sums of products of functions of u alone and v alone. He discusses a number of special cases. The computations are mostly in the classical spirit. The main new device is the following: if in the classical computation one has to extract a square root, to be taken with either a + or a − sign, one can take as "sign" an arbitrary measurable function $\epsilon(u,v)$, taking the values +1 and −1 only. The surfaces so obtained may no longer be differentiable. It can happen that two surfaces which are isometric, but cannot be deformed continuously into each other in the classical sense, can be so deformed in this larger class of surfaces.

H. Samelson (Ann Arbor, Mich.).

Buscheguennce, S. S. La géométrie d'un champ de vecteurs. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 155–158 (1945). [MF 16662]

La note actuelle reprend l'étude des champs de vecteurs par la méthode du repère mobile de É. Cartan. Le repère (J_1, J_2, J_3) attaché à chaque point M du champ est constitué par trois vecteurs unitaires deux à deux orthogonaux, J_3 ayant la direction du vecteur issu de M. Son déplacement est défini par les formes de Pfaff ω_0^α ($d\bar{M}=\omega_0^\alpha J_\alpha$) et par les quantités $p_\alpha, q_\alpha, r_\alpha$ qui définissent la rotation instantanée $\bar{\omega}=pJ_1+qJ_2+rJ_3$, $p=p_\alpha\omega_0^\alpha$, $q=\cdots$, $r=\cdots$.

L'auteur établit les relations entre les ω_0^α et les p_α donnant successivement: les trajectoires orthogonales du champ; la condition d'existence de surfaces orthogonales; les lignes de courbure (orthogonales aux plans $(J_3, \bar{\omega})$); la condition pour qu'une famille de surfaces orthogonales soit de Lamé; les lignes asymptotiques (dont les éléments $d\bar{M}$ sont situés dans les plans $(J_3, \bar{\omega})$ normalement à J_3, ou, ce qui revient au même, dont les binormales ont les directions des vecteurs J_3); les lignes géodésiques (dont les normales principales ont les directions des vecteurs J_3); les couples $d\bar{M}$ et $\delta\bar{M}$ unilatéralement conjugués (pour lesquels $\delta\bar{M}$ est orthogonal à J_3 et J_3+dJ_3) et bilatéralement conjugués ($d\bar{M}$ et $\delta\bar{M}$, normaux à J_3, divisent alors harmoniquement les directions asymptotiques). Il donne enfin les expressions des courbures moyenne, Gaussienne et totale et signale ceux des éléments envisagés qui reprennent leur signification ordinaire lorsque le champ considéré admet une famille de surfaces orthogonales. En ce qui concerne les lignes de courbure et les lignes asymptotiques, il est indiqué qu'elles ne se divisent harmoniquement que dans le cas où l'un des deux systèmes de courbes est orthogonal. *P. Vincensini* (Besançon).

Buscheguennce, S. S. La géométrie du champ de vecteurs. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 379–381 (1945). [MF 16649]

L'auteur poursuit l'étude commencée dans la note précédente. La courbure totale du champ (rapport des volumes des parallélépipèdes construits sur J_3, J_3+dJ_3, $J_3+\delta J_3$ et J_3, $d\bar{M}$, $\delta\bar{M}$) s'annule identiquement si, en chaque point, la direction d'une ligne de courbure est orthogonale à celle d'une ligne asymptotique, les directions de la deuxième ligne de courbure et de la deuxième ligne asymptotique étant confondues.

L'indétermination des lignes de courbure exige que les courbes du champ soient les trajectoires orthogonales d'une famille de sphères (ou de plans). Celle des lignes asymptotiques, exige que la congruence des courbes du champ soit celle des hélices lignes intégrales d'un champ de moments ou celle des trajectoires orthogonales d'une famille de plans à un paramètre. [Ce résultat se trouve déjà dans un mémoire de R. Garnier, Bull. Soc. Math. France 48, 106–108 (1920); il a été repris, avec le minimum d'hypothèses, par D. Lacombe, Revue Sci. 83, 167–169 (1945); ces Rev. 8, 49; voir aussi la deuxième analyse ci-dessus.]

L'auteur introduit et caractérise les champs de rotation (dont les vecteurs coupent un axe fixe) et les champs de complexes spéciaux (dont les vecteurs sont dans un complexe spécial). Il indique que, pour qu'avec les courbes d'un champ on puisse former deux familles de surfaces orthogonales, il suffit que le champ de l'une des bissectrices des asymptotiques admette une famille de surfaces orthogonales. Puis, par la considération des différentes familles de repères ayant J_3 en commun, il introduit six invariants liés par une relation et donne la signification de chacun d'eux.

P. Vincensini (Besançon).

Buscheguennce, S. S. La géométrie du champ de vecteurs. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 73–96 (1946). (Russian. French summary)

[The Russian spelling of the author's name is Byušgens.] This paper presents, in the notation of Cartan, some of the geometry of vector congruences in three-dimensional Euclidean space. Lines of curvature, geodesics and asymptotic

lines with respect to such congruences, and some of their properties, are derived. This study invites comparison with older work, such as that of A. Dell'Acqua [Atti Ist. Veneto Sci. Lett. Arti (8) 2, 245–252 (1899)], and on nonholonomic manifolds [Vranceanu, Gołab, and other writers after 1927]. *D. J. Struik* (Cambridge, Mass.).

Adelson-Welsky, G. Généralisation d'un théorème géométrique de M. Serge Bernstein. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 391–392 (1945).

The author proves that, if a surface defined by a continuous function $z=f(x, y)$ has generalized nonpositive curvature K, and if $\lim_{r\to\infty} |f(x, y)|/r=0$, $r^2=x^2+y^2$, then the surface is a cylinder with generators parallel to the (x, y)-plane. A surface $z=f(x, y)$ is said to have generalized nonpositive curvature if for every plane γ neither the set of points of γ satisfying $z<f(x, y)$ nor the set satisfying $z>f(x, y)$ contains a bounded component. Thus no differentiability assumptions on $f(x, y)$ are involved.
S. B. Myers (Ann Arbor, Mich.).

Krein, M. Sur un théorème de M. Vygodsky. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 447–450 (1946). (Russian. French summary)

The theorem [same Rec. N.S. **16(58)**, 73–80 (1945); these Rev. **7**, 75] states that a continuous closed spherical curve L is the tangent indicatrix of a closed curve, having a tangent at each point, if and only if L is contained in no hemisphere. Vygodsky gave a geometric proof; the author gives an analytic proof which also holds in n dimensions, $n\geqq 3$. *R. P. Boas, Jr.* (Providence, R. I.).

☆ The theorem is due to W. Fenchel [Jber. Deutsch. Math. Verein. 39, 183–186 (1930)]

Tietze, Heinrich. Über Orthogonalisierung, Kurventheorie und allgemeine Drehbewegung. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 127–129 (1944).

Announcement of results subsequently published in J. Reine Angew. Math. **186**, 116–128 (1944) [these Rev. **7**, 75].

Blaschke, Guillermo. Considerations on kinematics. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 179–182 (1946). (Spanish) [MF 16920]

Consider a fixed system A' of coordinate axes in three-dimensional Euclidean space and another system A which moves so that the position of A with respect to A' is periodic with respect to time. Let g be a straight line which is fixed with respect to A. Let P be an arbitrary point of g which moves so that its velocity (with respect to A') is always perpendicular to g. The problem considered in this note is that of finding the lines g for which the trajectories of the points P are closed curves. It is shown that in general these lines form a complex. The cases in which all of the ∞^4 lines possess the stated property appear to be trivial.
L. A. MacColl (New York, N. Y.).

Santaló, L. A. On a line complex related to a closed space curve. Math. Notae **6**, 45–56 (1946). (Spanish)

Let $O'A'B'C'$ be a fixed set of rectangular axes in Euclidean 3-space and let $OABC$ be a set of rectangular axes which moves so that it returns periodically to its initial position with respect to the fixed axes. Let g be a straight line which is fixed with respect to $OABC$. As $OABC$ moves, g describes a ruled surface S. Blaschke [see the preceding review] has studied the problem of selecting the line g so that the orthogonal trajectories of the generators of S are closed curves; he has found that in general all such lines g form a linear complex.

The author studies the complex further in the case in which $OABC$ moves so that the point O remains fixed and in the case in which $OABC$ is the fundamental moving trihedral of a closed curve Γ. In the first case he finds that the complex is generally singular, that is, consists of all lines which meet a certain fixed line. In the second case he obtains various theorems, including the following: in order that the complex shall be singular, it is necessary and sufficient that the total torsion of the curve Γ is zero.

Instead of requiring that the orthogonal trajectories of the generators of S shall be closed curves, we can require that the distance, measured along the generator, between two successive intersections of an orthogonal trajectory with a generator shall be a prescribed constant D. The author studies this problem, finds that the lines g satisfying the condition form a quadratic complex and discusses some properties of the complex. *L. A. MacColl.*

Santaló, L. A. A geometrical characterization for the affine differential invariants of a space curve. Bull. Amer. Math. Soc. **52**, 625–632 (1946).

The author obtains simple geometric characterizations for the affine differential invariants of a curve C by using at any point x_0 of C a quadric cone K determined by the parallels to five consecutive tangents to C, an osculating cubical parabola Q, and an osculating quadric cone K^* which has seven-point contact with C. For instance, let A_1 and A_2 be the points at which Q intersects K. The osculating plane of Q at either A_1 or A_2 determines with the affine fundamental trihedral at x_0 a tetrahedron whose volume V is related to the affine curvature k by $V=\frac{4}{3}k^{-4}$.
A. Schwartz (State College, Pa.).

Santaló, L. A. Some properties of skew curves in projective differential geometry. Actas Acad. Ci. Lima **8**, 203–216 (1945). (Spanish)

Cet article traite de certaines surfaces développables coutenant une courbe donnée de l'espace projectif à trois dimensions. Le paramètre définissant les différents points de la courbe $X(u)$ est l'arc projectif u, et le repère attaché à un point quelconque de la courbe est le tétraèdre fondamental (X, X', X'', X''') dont les faces issues de X sont le plan osculateur, le plan normal et le plan rectifiant projectifs.

L'auteur forme d'abord la condition pour qu'une surface reglée dont les génératrices s'appuient sur la courbe (X) soit développable, puis la condition supplémentaire qui exprime que la développable est un cône. Il applique les résultats obtenus aux cas où les développables sont formées de droits situées soit dans le plan rectifiant soit dans le plan normal projectif. Dans le cas du plan rectifiant on trouve une surface développable, qui devient un cône si la deuxième courbure projective (s) de la courbe (X) est nulle. Dans le cas du plan normal projectif, on obtient ∞^1 développables, fournies par une équation de Riccati dépendant uniquement de la première courbure projective (r) de la courbe donnée. Les courbes pour lesquelles l'une des surfaces développables précédentes est un cône sont caractérisées par une relation entre r et s présentée sous une forme particulièrement simple.

L'auteur termine par la recherche des courbes pour lesquelles il existe des développables dont les génératrices sont projectivement fixes par rapport au tétraèdre fondamental. Ces courbes sont caractérisées par le fait que leur première courbure r est constant; il existe ∞^1 développables associées, et les génératrices de ces développables issues d'un même point de la courbe forment un cône du second degré tangent

au plan osculateur suivant la tangente. Le cas où parmi les développables précédents figurent des cônes (il y en a alors 1, 2, 3 ou 4) exige que les deux courbures r et s soient séparément constantes, et ramène aux courbes W (courbes anharmoniques) invariantes par un sous-groupe à un paramètre du groupe projectif. *P. Vincensini* (Besançon).

Abramescu, N. Sur la développée affine d'une courbe plane. Mathematica, Timişoara **22**, 69–73 (1946).

The determination of a curve, the center of whose osculating conics describes a parabola, leads to a Riccati differential equation. *V. G. Grove* (East Lansing, Mich.).

Stašek, Pavel. Über die Flächen, deren Krümmungslinien sich auf eine Ebene in gegebene Kurven zentral projizieren. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1940, 13 pp. (1940). (German. Czech summary) [MF 16122]

Consider a surface σ and a fixed point S on the z-axis. By projecting, through the point S and onto the (x, y)-plane, the points on the lines of curvature of σ, one obtains two families of curves in the (x, y)-plane. The author's problem is to determine all surfaces for which the two congruences of (x, y)-plane curves are orthogonal. It is shown that the solution of this problem depends upon the solution of a system consisting of a linear second order partial differential equation and a nonlinear first order partial differential equation. Because of the complexity of the system, the author considers only two special cases: (1) the congruences consist of circles with origin as center and radial lines; (2) the system consists of two families of confocal conics. In each of these cases, the surfaces are determined. In the second part of the paper, the author applies tensor calculus to the problem in an effort to determine the relation between the fundamental tensors of the surface and the net in the (x, y)-plane. However, due to an error in the equations defining the projectivity, the author solves the problem for the case where the center of projectivity undergoes some undetermined variation. *N. Coburn.*

Marcus, F. Sur les réseaux avec les invariants absolus constants. Bull. Math. Soc. Roumaine Sci. **46**, 135–140 (1944). [MF 16516]

L'auteur désigne par h et k les invariants de Darboux $h=a_u+ab-c$, $k=b_v+ab-c$, de l'équation de Laplace $x_{uv}+ax_u+bx_v+cx=0$ qui correspond à un réseau (x) de l'espace projectif à n dimensions. Pour les transformations $x=\lambda(u, v)X$, $u=\phi(\alpha)$, $v=\psi(\beta)$ le rapport $m=h/k$ est un invariant absolu. L'auteur montre que les invariants m_n des réseaux transformés de Laplace se calculent par une formule simple en fonction de m (ou m_0), m_1 et n (n peut être positif ou négatif). On en déduit qu'une suite de Laplace aux m_n constants ne peut être périodique qu'avec des invariants égaux à un. On donne aussi les conditions pour que la suite soit terminée à (x_n) selon le cas de Laplace ou le cas de Goursat. *M. Decuyper* (Lille).

Marcus, F. Sur quelques congruences. Mathematica, Timişoara **22**, 204–205 (1946).

This note adds a few results, stated without proof, to a previous paper by the author [same journal **20**, 23–28 (1944); these Rev. **7**, 32]. Some of these may be stated as follows. Two opposite sides of a skew quadrilateral of Laplace generate congruences of Waelsch if the congruences generated by the other two sides are stratifiable in one sense. The four sides of a skew quadrilateral of Laplace generate congruences of Waelsch if the pairs of opposite sides generate stratifiable pairs of congruences. *V. G. Grove* (East Lansing, Mich.).

Mihăilescu, Tiberiu. Sur les congruences de Ribaucour-Darboux. Mathematica, Timişoara **22**, 51–56 (1946).

Given a conjugate net N on a quadric Q, there exists a congruence Δ whose developables intersect Q in N and hence also in a second conjugate net N'. The nets N and N' are said to be related by a transformation of Ribaucour and Δ is said to be a congruence of Ribaucour-Darboux. Such a congruence is known to depend on two arbitrary functions of one variable. This paper studies such congruences using Pfaffian systems in involution. Some of the results may be stated in the following manner. A general congruence depends on two arbitrary functions of two variables. The congruences of Ribaucour-Darboux form a special class depending on one arbitrary function of two variables. Let Δ be such a congruence, Q the associated quadric and Δ' the congruence of the polar lines of the lines of Δ with respect to Q. Then Δ and Δ' are stratifiable in some sense; conversely, if Δ and its reciprocal Δ' with respect to a quadric Q are stratifiable in one sense, then one of Δ, Δ' is a congruence of Ribaucour-Darboux. In order that the focal points of the lines of a congruence Δ' be conjugate with respect to a quadric Q, it is necessary and sufficient that the polar congruence Δ of Δ' with respect to Q be a congruence of Ribaucour-Darboux with Q as the associated quadric. *V. G. Grove* (East Lansing, Mich.).

Cartan, Élie. Sur un problème de géométrie différentielle projective. Ann. Sci. École Norm. Sup. (3) **62**, 205–231 (1945).

The following problem is posed. Being given a real projective space of r dimensions ($r\geqq 3$), at each point M of which is attached according to a determined law a hyperplane P (a linear variety of p dimensions, $p=r-1$), a field of contact elements (M, P) is thus defined. A curve will be called a curve of the field if at each of its points M its tangent line lies in the hyperplane P associated with this point. Determine the fields which possess the property that every curve of the field has at each of its points M contact of the second order with the hyperplane P associated with this point; or simply, determine the fields whose asymptotic lines are indeterminate. The author calls a field of plane elements of p dimensions having the above property a field (C). The following theorem and corollary are proved as preliminaries. Being given a field (C) of plane elements of p dimensions in the space of r dimensions all of the lines which pass through any point M and lie in the plane variety P associated with this point are lines in the field. If a point N lies in the plane associated with the point M, reciprocally the point M lies in the plane associated with the point N.

The author determines the fields (C) first in the ordinary projective space of 3 dimensions and then in the projective space of r dimensions. The fields (C) are found to be of two classes. If the equation of Pfaff, which defines in a domain D of the space the field (C) of contact elements sought, is completely integrable, the field is obtained in considering a one-parameter family of hyperplanes such that one and only one plane of the family passes through each point of the domain; the hyperplane associated with a point M is then the hyperplane of the family passing through this point. The solutions of this class cannot in general be extended to all points of space. If the equation of Pfaff is

not completely integrable, the hyperplane to associate with a point M of the domain is the focal hyperplane of the point M with respect to a fixed linear complex. The solutions of this class are extensible to all of the space. To obtain the general solution of this class the author employs his method of the "repère mobile." With each point A_0 of the domain D there is associated a "repère" formed by $r+1$ analytic points $A_0, \cdots, A_r$. The differential equations of the problem assume the forms of the systems of Pfaff $\omega_{2i-1}^r=\omega^{2i}$, $\omega_{2i}^r=-\omega^{2i-1}$, $\omega_\alpha^i=0$, $i=1, \cdots, p$; $\alpha=2p+1, \cdots, r$, in which two cases are distinguished according as $p=1$ or $p>1$, and whose terms are the coefficients of the equations of displacement $dA_i=\omega_i^k A_k$, $\omega_0^k=\omega^k$, $i, k=0, 1, \cdots, r$. The general solution is found to depend on $2(p+1)r-(2p^2+3p+2)$ arbitrary constants. The solutions of this class are projectively equivalent to one another with respect to a projective group with $r^2-2pr+2p^2+3p+2$ parameters.

The author points out that the problem of determining the fields (C) may be generalized by associating at each point M of the projective space of r dimensions a linear variety P with $p<r-1$ dimensions passing through this point. He determines the solutions of this problem for the particular case of the field (C) of contact elements of two dimensions in a projective space of four dimensions. The curves of such a field are those which satisfy two independent equations of Pfaff. There are three categories of solutions, the first two of which are of local character and cannot in general be extended to the entire space. The third class of solutions arises in case no linear combination of the two equations of Pfaff is completely integrable. The principal result here is that the plane associated with the point M in the field (C) is the intersection of two hyperplanes which are focal hyperplanes of the point M with respect to two fixed linear complexes. *P. O. Bell.*

Choquet, Gustave. Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques. Bull. Sci. Math. (2) **69**, 156–165 (1945).

A harmonic mapping of a plane domain δ on a plane domain Δ is a homeomorphism between δ and Δ having the representation $X=X(x, y)$, $Y=Y(x, y)$, where $X(x, y)$ and $Y(x, y)$ are harmonic in δ. This type of mapping may be considered to be a generalization of a conformal mapping of δ on Δ. The author's main result may be stated as follows. Let δ be the domain interior to a Jordan curve γ and let Δ be a bounded convex domain with frontier Γ. Then every homeomorphism between γ and Γ can be continuously extended by a harmonic mapping of δ on Δ. The hypothesis of convexity of Δ cannot be dropped. In addition, several interesting problems are posed, for example, is there an analogous theorem in three-space? *M. O. Reade.*

Bidal, Pierre, et de Rham, Georges. Les formes différentielles harmoniques. Comment. Math. Helv. **19**, 1–49 (1946).

The definition of harmonic forms as given by the reviewer [The Theory and Applications of Harmonic Integrals, Cambridge University Press, 1941; these Rev. 2, 296] is modified as follows. Let M be an orientable manifold carrying a Riemannian metric $g_{ij}dx^i dx^j$, P a p-fold differential form. The exterior derivative of P (a $(p+1)$-form) is denoted by dP. The dual form

$$\sqrt{g}\,\epsilon_{j_1\cdots j_p i_1\cdots i_{n-p}} g^{j_1 k_1} \cdots g^{j_p k_p} P_{k_1\cdots k_p} dx^{i_1} \cdots dx^{i_{n-p}}$$

is denoted by P^*. The co-derivative of P (a $(p-1)$-form) is defined by $\delta P=(dP^*)^*$. The usual definition of a harmonic p-form is one which satisfies the two equations $dP=0$, $\delta P=0$. The authors define P to be a harmonic form when $\Delta P=(-1)^{np}\delta dP+(-1)^{np+n}d\delta P$. While this definition differs materially from the usual one for forms considered locally, when the forms are harmonic everywhere on M the two definitions are shown to be equivalent. One advantage of the new definition is that, whereas Hodge [loc. cit., p. 131] reduced the proof of the fundamental existence theorem to the problem of solving the differential equation $\delta dP=f$ when $\delta f=0$, which is not elliptic, since the equation $\delta dP=0$ has an infinite number of linearly independent solutions, the authors base their proof on the solution of the equation $\Delta P=f$ which is strictly elliptic.

The paper gives a systemic account of harmonic forms on a Riemannian manifold ab initio and establishes the fundamental existence theorem by the parametrix method used by Hodge adopting the same parametrix as Hodge. Both the general account of harmonic forms and the proof of the existence theorem are notably simpler than in Hodge's work, though the methods are similar throughout.

W. V. D. Hodge (Cambridge, England).

Norden, A. On Weyl's projectively Euclidean space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 307–308 (1945). [MF 16655]

A geometric determination of two-dimensional projectively Euclidean spaces is given by means of a projective interpretation of these spaces proposed by Bortolotti [Ann. Mat. Pura Appl. (4) **11**, 111–134 (1932)]. According to Bortolotti, a certain dual correspondence may be correlated to every projectively Euclidean space. In the present case, for the construction of this dual correspondence it is necessary to define two arbitrary "absolute" curves in the projective plane and to correlate the point of intersection of their tangents with the straight line joining the points of contact. *A. Fialkow* (New York, N. Y.).

Norden, A. The affine connectivity on the surfaces of a projective and conformal space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 539–541 (1945). [MF 16644]

A surface X_m of m dimensions embedded in a projective space P_n is called normalized if with every point Q of X_m are connected (1) a P_{n-m} containing Q not identical with the tangent plane T_m and (2) a P_{m-1} belonging to T_m not containing Q. These planes define an affine connection in X_m. In case there exists a dual correspondence in P_n, two different connections, which are dual to each other, are defined on a normalized hypersurface. A surface X_m in the Möbius space M_n is called normalized if at every point Q of X_m there is defined a sphere S of $n-m$ dimensions passing through Q and totally normal to X_m. Then a connection again exists. For $m=n$ it is sufficient to connect to each point another point. If this correspondence is an inversion with respect to a hypersphere one obtains a space of constant curvature. *J. Haantjes* (Amsterdam).

Norden, A. On pairs of conjugate parallel translations in n-dimensional spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 625–628 (1945).

Two affine connections G and Γ with symmetrical coefficients g_{ij}^k and Γ_{ij}^k are said to be conjugate to each other with respect to a given tensor b_{ij} if with every direction v^h that can be displaced parallel to itself in G a hyperplane conjugate to v^h with respect to b_{ij} is associated which can be

transported parallel to itself in Γ. The author finds the conditions of conjugateness and considers some special cases.
J. Haantjes (Amsterdam).

Norden, A. Espace à connexion affine dont l'intégrale des géodésiques est exprimée par l'équation $\frac{Adu+Bdv}{Cdu+Ddv}=k$. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 125–138 (1946). (Russian. French summary) [MF 16680]

The author calls the double family of integral curves of the equation of the title an anharmonic field, while the one-parameter family corresponding to a fixed k is called a cluster. The problem considered is that of a two-dimensional affine connection whose paths form an anharmonic field. The problem is not solved except for a very special connection. If, however, the connection is required to have two different first integrals of the above fractional type, then it is shown the connection is necessarily that of Weyl and the paths in this case are maps of generalized parabolas in the plane (curvilinear directrix). Finally, a Riemann anharmonic metric is shown to have the form

$$ds^2=(u_x^2+u_y^2)(dx^2+dy^2),$$

where $y(u_{xx}+u_{yy})+u_y=0$, thus showing that u is a cylindrical harmonic function. Thus the metric of an anharmonic space may be expressed in terms of Legendre polynomials, Bessel functions, etc. The geodesics of such a space may be obtained by quadratures.
M. S. Knebelman.

Norden, A. La géométrie généralisée de l'espace reglé à deux dimensions. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 139–152 (1946). (Russian. French summary) [MF 16681]

The paper gives an ingenious geometrical interpretation of differential invariants in S_2, the coordinates defining a "straight line." The word "generalized" in the title is, however, misleading. The paper is really concerned with an equiaffine space admitting a covariant vector field with vanishing covariant derivative; this implies that this vector φ_i is a gradient of a function φ. Also, equiaffinity implies the existence of a skew symmetric bivector ϵ_{ij} with vanishing covariant derivatives, but this bivector is not unique. Thus the structure of S_2 is really an equiaffine connection, a potential function φ and a fundamental bivector $\epsilon_{ij}=-\epsilon_{ji}$, $\nabla_k\epsilon_{ij}=0$. A point then is a contravariant vector, linearly dependent vectors defining the same point. By means of this structure the author defines the angle between two lines, the distance between two points on the same straight line, the Crofton measure of a region of lines (an integral invariant), the arc length of a curve and its radius of curvature. The scalar curvature of S_2 is defined analogously to the Gaussian curvature as defined by the spherical indicatrix of the surface normal. If this curvature for S_2 has the value -1, the geometry then coincides with that of a Euclidean plane of lines.
M. S. Knebelman.

Blank, J. Das Linienelement der Fläche in der Kugelgeometrie. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnil Zbirnik **2**, 3–7 (1940). (Russian. German summary)

The author's summary is as follows. Der Begriff des Linienelements der Fläche in der Kugelgeometrie wird am besten eingeführt ausgehend von der Erklärung, welche ganz analog der Terracini'schen geometrischen Interpretation des projectiven Linienelements ist. Der Verfasser bemerkt zum Schluss, dass man auch in der Kugelgeometrie von den Flächenstreifen sprechen kann, welche durch das Verschwinden der Variation des Linienelements erklärt werden.
M. S. Knebelman (Pullman, Wash.).

Rosenfeld, B. A. Die metrische Geometrien des Kugelraumes. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 59–82 (1944). (Russian. German summary) [MF 15195]

The totality of spheres of a 3-space of constant positive curvature is mapped following Lie (who, however, considered a flat 3-space) on a four-dimensional quadric L of a projective 5-space. Lie himself considered the geometry of L corresponding to a 15-parameter group of projective transformations of the 5-space which leave L invariant. Here various "metric" geometries of L are introduced which correspond to different 10-parameter groups of "rigid motions" (each of them assigns an invariant, "distance," to every pair of spheres). This is interpreted in terms of a "geography" of L: L is covered by a set of ∞^3 meridians and ∞^1 (three-dimensional) parallels; each of the metric geometries may be characterized by an absolute which is a four-dimensional quadric tangent to L "along" some parallel; the corresponding group leaves the absolute as well as L invariant; in terms of the spheres of the original 3-space one may say that the group transforms spheres of a certain radius into spheres of the same radius. Two of these groups have been considered before (or rather, analogous groups corresponding to spheres in flat space), one by Möbius and the other by Laguerre; they leave invariant spheres of radius zero and $\pi/2$, respectively. A third group of rigid transformations which the author singles out for special consideration leaves invariant autopolar spheres (spheres of radius $\pi/4$); this group has no counterpart in the case of flat space. All the metric groups are subgroups of an 11-parameter group whose geometry is also investigated. To curves of L correspond, of course, one-parameter families of spheres in the original 3-space; the author studies such families and their envelopes corresponding to geodesics, asymptotic lines, etc. Some of these envelopes generalize Dupin's cyclides.

Misprints, some of them disturbing, complicate the reading of the paper.
G. Y. Rainich (Ann Arbor, Mich.).

Dubnov, J., et Sabyrov, M. Les tenseurs fondamentaux dans la théorie métrique des congruences de sphères. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 615–617 (1945).

A metrical theory of congruences of spheres was developed by Bianchi [Lezioni di Geometria Differenziale, vol. 2, Bologna, 1927, chap. 19]. The authors define two tensors, corresponding to the two quadratic forms of Bianchi. If the radius ρ and these two tensors satisfy certain differential equations it is proved that these quantities determine the congruence except for translations and rotations. After eliminating ρ it is proved that if the tensors satisfy certain differential equations they determine the congruence except for translations, rotations and dilatations.

Between the two tensors of Bianchi and the first and second fundamental tensor of one of the enveloping surfaces there exists a reciprocity, but only an algebraical one, the tensors of Bianchi not satisfying the conditions of Gauss-Codazzi. An algebraical relation is given between these two sets of tensors. As an example the congruences of Ribaucour are dealt with.
J. A. Schouten (Epe).

Dubnov, Ya. Complete system of invariants of two affinors in centro-affine space of two or three dimensions. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 5, 250–270 (1941). (Russian) [MF 15614]

An expression linear in each of n affinors A_i giving a simultaneous invariant of these affinors can be used to build simultaneous invariants of two affinors Φ and Ψ but of higher degree by replacing the A_i by polynomials in Φ and Ψ. In the first part of the paper a set S of invariants is called complete if all invariants of the type described above are rationally expressed in terms of the invariants in S. Using traces of power products of Φ and Ψ complete sets are constructed for two dimensions (consisting of 5 invariants) and for three dimensions (12 invariants, of which two are roots of a quadratic whose coefficients are rational in the other ten). In the second part conditions for equivalence of two pairs of affinors in n-space are discussed in terms of their characteristic numbers and relative positions of their axes (invariant directions). For $n=2, 3$ these conditions yield the result that the sets of invariants introduced in the first part are complete also in the sense that two pairs of affinors are equivalent if all the invariants of the set have equal values (for $n=3$, if only the first ten invariants have the same values for the two pairs of affinors these pairs are "conjugate"). In these cases the relative positions of the axes are interpreted in terms of projective geometry.

G. Y. Rainich (Ann Arbor, Mich.).

De Cicco, John. Differential geometry in the Kasner plane. Amer. Math. Monthly 53, 305–313 (1946). [MF 16721]

A horn angle is the configuration formed by two curves through a common point in a common direction; a simple horn-set is the totality of smooth curves with the common point and direction. The Kasner measure of horn angle is defined differently for each category (order of contact); in each case it is invariant under the group of conformal transformations.

The measure of a horn angle of the first category (simple contact) is a function of the curvature, and of the rate of change of the curvature with respect to arc length, of the two curves forming the angle:

$$M_{12}=(\gamma_2-\gamma_1)^2/(d\gamma_2/ds_2-d\gamma_1/ds_1).$$

The Kasner plane is introduced for the study of the conformal geometry of horn angles of the first category: relative to a given simple horn-set S, the x-coordinate of the Kasner plane is the curvature γ of a curve of S and the y coordinate is $d\gamma/ds$. The group of conformal transformations, operating on the curves of a simple horn-set, induces a three-parameter group of affine transformations in the Kasner plane. The author begins the study of the differential geometry of this group. Items discussed include arc length, arc curvature, curvature, evolutes and involutes.

E. F. Beckenbach (Los Angeles, Calif.).

Kasner, Edward and DeCicco, John. Geometry of the Fourier heat equation. Trans. Amer. Math. Soc. 60, 119–132 (1946).

If a homogeneous and isotropic region of space is heated by conduction the temperature $v=\phi(x, y, z, t)$ satisfies the equation $\phi_{xx}+\phi_{yy}+\phi_{zz}=\phi_t$. The ∞^2 surfaces $\phi(x, y, z, t)=v$, the parameters being v and t, are called the heat surfaces. It is proved that there are no systems of ∞^2 planes which make up a heat family. If all the heat surfaces are planes, the family degenerates into ∞^1 and is a pencil in the real domain. In the real domain there are no systems of ∞^2 spheres which form a family and, if all the heat surfaces are spheres, the family degenerates into ∞^1 concentric spheres. In the imaginary domain it is found that the only heat families of ∞^2 spheres are those whose centers describe a minimal straight line. The results are generalized to Euclidean spaces of $n>3$ dimensions. The only point transformations of Euclidean space of $n\geqq 3$ dimensions which convert every isothermal system of hypersurfaces into an isothermal system are those of the similitude group. This result is quite different from that for $n=2$, in which case it is the conformal group. Furthermore, it is shown that Lie's characterization of isothermal families in the plane is not valid in general for $n\geqq 3$. *J. Haantjes* (Amsterdam).

Haimovici, M. Sur les systèmes homographiques de courbes dans l'espace à n dimensions. Mathematica, Timişoara 22, 89–90 (1946).

G. Thomsen [Abh. Math. Sem. Hamburgischen Univ. 8, 115–122 (1930)] defined and studied homographic systems of curves in the plane. The present writer shows that with such a homographic system in n-dimensions an affine connection of distant parallelism can be associated under which the curves of a congruence of the family are parallel geodesics. The connection depends on an arbitrary Pfaffian.

J. L. Vanderslice (College Park, Md.).

Haimovici, Adolf. Sur une famille de surfaces généralisant les quadriques de révolution. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 46–61 (1946).

In a Riemann space a vector field is "concurrent" if $v_{,\mu}{}^{\lambda}+\delta_{\mu}{}^{\lambda}=0$. Let such a space contain a hypersurface along which there exist two concurrent vector fields which at any point are symmetrical to the normal to the hypersurface at that point. Such a hypersurface is a generalization of a conic or a quadric of revolution. The author deals with a Riemann space of three dimensions and assumes that the coordinates are referred to a triply orthogonal system of hypersurfaces. Then $ds^2=g_{11}(du^1)^2+g_{22}(du^2)^2+g_{33}(du^3)^2$. He then develops general expressions for the $g_{\alpha\alpha}$ under the condition that these hypersurfaces have the "conic" property described above. It is furthermore proved that the asymptotic lines of these hypersurfaces are geodesics. The similar problem for "conics" in a two-dimensional surface was solved by A. Myller [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 17, 112–115 (1933)]. *C. B. Allendoerfer.*

Čahtauri, A. I. Geometry connected with a correlative transformation. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 13, 101–137 (1944). (Russian. Georgian summary) [MF 14622]

The paper deals with the relationship between the algebraic and analytic approaches to geometry. First, following Bortolotti [Ann. Mat. Pura Appl. (4) 9, 111–134 (1932)] and Norden [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 2–3, 229–268 (1935); unpublished dissertation] a method of introducing an affine connection into a space with a given correspondence between points and hyperplanes is presented; this connection, which is projectively Euclidean, is then specialized for the case when the correspondence is given by a nonsymmetric matrix (the case of a polarity, when the matrix is symmetric, has been studied before, and is here considered as a special case along with some other special cases); the resulting space is called correlative. The case $n=2$, requiring special consideration, is treated in chapter II. In the last

chapter the author follows a portion of Norden's paper mentioned above, extending to the case of a pair of surfaces in correlative space some of the concepts and results discussed in that paper for projective space.
G. Y. Rainich (Ann Arbor, Mich.).

Petrescu, St. Quelques propriétés des espaces non holonomes A_n^m, considérés dans un espace A_n, à connexion affine, admettant un parallélisme absolu. Bull. Math. Soc. Roumaine Sci. **46**, 145–154 (1944). [MF 16518]

Dans un L_n à connexion affine, admettant un parallélisme absolu, c'est-à-dire un L_n qui admette n vecteurs covariants constants, les composantes de la connexion par rapport aux congruences de ce parallélisme sont nulles. L'auteur prouve que la connexion de Schouten de l'espace non holonome L_n^m, défini par des congruences de ce parallélisme, a toutes ses composantes nulles et que l'espace L_n^m est une variété géodésique.
J. Haantjes (Amsterdam).

Strubecker, Karl. Differentialgeometrie des isotropen Raumes. IV. Theorie der flächentreuen Abbildungen der Ebene. Math. Z. **50**, 1–92 (1944). [MF 15842]

[Parts I, II and III appeared in Akad. Wiss. Wien. S.-B. IIa **150**, 1–53 (1941); Math. Z. **47**, 743–777; **48**, 369–427 (1942); cf. these Rev. **5**, 109; **7**, 530.] Equiareal transformations of the sphere have been investigated by means of their close relationship to the theory of surfaces in elliptic space [see, for example, G. Fubini, Ann. Scuola Norm. Super. Pisa **9**, no. 1 (1904) or L. Bianchi, Lezioni di Geometria Differenziale, 3d ed., v. 2, Pisa, 1923]. This paper uses a similar method for the investigation of equiareal transformations of the plane by means of their relationship to the theory of surfaces in isotropic three-dimensional space. The underlying principle is the possibility of mapping the plane elements of a surface (more generally, a two-dimensional set of plane elements in the sense of Lie) on the point pairs of a plane by means of the two Clifford parallelisms. Such a "paratactic" mapping is equiareal, and every equiareal mapping can be considered as a "paratactic" mapping. The author proceeds to a classification of equiareal transformations and discusses many special cases. Curves in isotropic space correspond to transformations of which the locus of the middle points is one-dimensional; other interesting transformations correspond to minimal surfaces or to surfaces of constant relative curvature $K=+1$. These last mappings have a close relation to the theory of complex analytic curves. There is also a very readable summary of the differential geometry of curves and surfaces in isotropic space.
D. J. Struik.

Davies, E. T. The geometry of a multiple integral. J. London Math. Soc. **20**, 163–170 (1945).

In an n-dimensional point manifold a geometry is based on a k-fold integral taken over a general k-dimensional submanifold; when $k=1$ we have Finsler geometry and when $k=n-1$ we have Cartan's geometry based on the idea of area. The integral gives rise to a scalar L, a function of coordinates and an arbitrary k-vector, and the problem is to derive fundamental metric tensors, connexions and curvature tensors from L. For formal simplicity the author considers only the cases $k=2$ and $k=n-2$, and later confines himself to $n=5$. The scalar L is now a function of coordinates x and a bivector density u which is contravariant of weight p when $k=2$ and covariant of weight $-p$ when $k=n-2$. This bivector is analogous to the element of support in earlier work; corresponding to the earlier fundamental unit vectors there are now fundamental unit bivectors l^{ij} and l_{ij}. The fundamental metric tensor is $g_{ij}(x, u)$ and is defined in terms of L and its derivatives with respect to the components of u.

Absolute derivation is introduced, the absolute differential of a vector X^i being, for example,

$$DX^i=dX^i+\Gamma^i_{jh}X^j+C^i_{jkl}X^jdu^{kl}$$

in the case $k=2$. Here Γ^h_{jh} and C^i_{jkl} are expressible in terms of g_{ij} and its derivatives by means of certain conventions; the expression for C^i_{jkl} is given. Sufficient equations for determining Γ^i_{jh} are given but not its actual expression, which is too complicated. Absolute derivatives $T|_i$ are now defined. Tensor derivatives $T\|_{ij}$ have already been defined and all the various alternation formulae with these two derivatives are now calculated. Formulae such as for $X^i|_h|_k-X^i|_k|_h$ give rise to a tensor $\mathring{R}^m_{mjh}$, which in turn leads to a curvature tensor R^h_{ijk}.
A. G. Walker (Liverpool).

Varga, O. Zur Begründung der Minkowskischen Geometrie. Acta Univ. Szeged. Sect. Sci. Math. **10**, 149–163 (1943). [MF 16742]

In an E^n with affine coordinates $x^1, \cdots, x^n$ let $L(dx)=L(dx^1, \cdots, dx^n)$ determine a Minkowski metric. If $g_{ik}(\dot{x})=\frac{1}{2}\partial^2L^2(\dot{x})/\partial\dot{x}_i\partial\dot{x}_k$, then $g_{ik}(\dot{x})dx^idx^k$ is called the osculating Euclidean metric $(\dot{x})$ of $L(dx)$ in the direction $\dot{x}$. For each osculating metric $(\dot{x})$ determine one coordinate system with unit vectors $e^{(i)}(\dot{x})$ such that $e_s^{(i)}e_s^{(k)}=g_{ik}(\dot{x})$ and the $e_s^{(i)}$ are continuously differentiable in the $\dot{x}^l$. Vectors with respect to a fixed direction $\dot{x}$ can then be defined and compared in the obvious (Euclidean) way. Let a vector field $v(x, \dot{x})$ be given. Then comparison of vectors $v(x, \dot{x})$ and $v(x+dx, \dot{x}+d\dot{x})$ is defined by $Dv^s=dv^s+C^s_{kl}d\dot{x}^l$, where the $v^i(x, \dot{x})$ are the components of $v(x, \dot{x})$ with respect to the local coordinate system $e_k^{(i)}(\dot{x})$ and the C^s_{kl} are determined by the relations $\partial e_i^{(s)}/\partial\dot{x}^k=C^r_{sk}e_i^{(r)}$, $C^l_{ir}g_{lk}=\frac{1}{4}\partial^3L^2(\dot{x})/\partial\dot{x}_i\partial\dot{x}_k\partial\dot{x}_r$. Then the formulas are derived which correspond to these when the Cartesian coordinates x^i in E^n are replaced by arbitrary curvilinear coordinates. In analogy to Cartan's treatment of Riemannian geometry, it is shown that the new formulas can be used as a base for developing a general theory of Finsler spaces.
H. Busemann.

Berwald, L. Ueber die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen. Nederl. Akad. Wetensch., Proc. **49**, 642–647 = Indagationes Math. **8**, 401–406 (1946).

Recherche des formules permettant de passer de la connexion attachée à un espace de Finsler par E. Noether et par l'auteur [Math. Z. **25**, 37–60 (1926)] à celle donnée par É. Cartan [Les espaces de Finsler, Actual. Sci. Ind., no. 79, Hermann, Paris, 1934]. L'auteur compare en particulier les tenseurs de courbure des deux connexions et étudie le cas où ces deux connexions coincident.
A. Lichnerowicz (Strasbourg).

van der Kulk, W. The $(1, \infty^t)$-contact transformations of the E_m's in X_n. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 421–428 (1943). (Dutch. German, English and French summaries) [MF 15780]

All contact transformations are determined which map all E_m of an X_n on a system of E_m in another space X_N by a $(1, \infty^t)$-correspondence. Here $m+1<n$, $t\geqq 0$. Besides transformations which map all E_m of an X_n on all E_m of a union of elements in X_N and extended point transformations of X_n on an X_p in X_N $(m<p\leqq n)$, only one type of such con-

tact transformations exists, mapping all E_m of X_n on all E_m of an X_{m+1} in an X_N. In this case $t=(m+1)(n-m+1)$. For $m+1=n$ we have the case studied by Lie [Theorie der Transformationsgruppen, v. 2, Leipzig, 1890, p. 108, footnote 1]; for the case $m+1<n$ there is a theorem by A. V. Bäcklund [Math. Ann. 9, 297–320 (1876)] which deals with (1, 1) correspondences in the case $m+1<n$.

D. J. Struik (Cambridge, Mass.).

Lee, H. C. Contact transformations. Duke Math. J. **13**, 161–170 (1946).

The author constructs a tensor analysis of contact transformations. Essentially, the paper furnishes a formal scheme for treating nonhomogeneous contact transformations by a tensor analysis. Let x^0, x^i, p_j, $i, j=1, \cdots, n$, be $2n+1$ independent variables; let $\bar{x}^0$, $\bar{x}^i$, $\bar{p}_j$ be $2n+1$ functions of the former variables. The contact transformation may be written as (1) $d\bar{x}^0-\bar{p}_i d\bar{x}^i=\rho(dx^0-p_i dx^i)$, where ρ is the factor of the transformation. By writing (2) $p_i=x^{n+i}$, $\bar{p}_i=\bar{x}^{n+i}$, the equation (1) takes on the invariant form (3) $\bar{q}_a d\bar{x}^a=\rho q_a dx^a$, where the q_a's are functions of the $2n+1$ independent variables x^j ($j=0, \cdots, 2n$); see (1). Most of the paper consists in determining tensors which can be constructed from the density vector q_a and, in particular, those tensors which characterise the contact transformation. *N. Coburn.*

***Brillouin, Léon. Les Tenseurs en Mécanique et en Élasticité.** Dover Publications, New York, N. Y., 1946. xi+364 pp. $3.75.

[This is a reprint of the edition of 1938, published by Masson et Cie., Paris.] This text has two purposes: (1) to acquaint the tensor analyst with the application of his subject to classical and modern physics; (2) to acquaint the physicist with tensor analysis and show the value of this subject as a tool in physics. The following topics are treated: tensor analysis, classical and wave mechanics, elasticity and the quantum theory of solids.

The first part is concerned with tensor analysis. Concepts such as vectors, tensors (symmetric and antisymmetric), matrices of transformations, affine and metric spaces are introduced and their meanings are illustrated by use of geometry and physics. For instance, a physical illustration of affine spaces is given by means of the thermodynamic plane. The rules for forming tensors (sum, product, etc.) bivectors, trivectors, pseudo-scalars and pseudo-tensors are carefully explained. Gradient, curl and divergence are discussed. By use of invariant ideas, the author is able to furnish a relatively simple proof of Stokes's theorem and to determine its generalization to higher dimensional spaces. Following Weyl's approach, he develops the theory of affine spaces; by introducing a metric tensor, he obtains Riemannian spaces. Parallelism, the Riemann-Christoffel and Ricci tensors, the Bianchi identities, geodesics, geodesic and normal coordinates are introduced and discussed. Of special importance to the physicist are the formulas which the author develops for the gradient, divergence and Laplacian in triply orthogonal coordinate systems and the explanation which he offers of the relations between vectors in tensor notation and vectors in the ordinary vector analysis notation. The latter are the components of a vector referred to a local Euclidean tangent space.

The second portion of the book is devoted to the applications of tensor analysis to classical and wave mechanics. It is shown that d'Alembert's principle has a simple geometric meaning: the projection of the sum of the external and inertial forces, on a subspace perpendicular to the subspace determined by the constraints, vanishes. By use of the kinetic energy, a metric tensor is defined in the space whose variables are the generalized coordinates of a dynamical system. The Lagrange equations for the case of no external forces are shown to imply that the path of the system is a geodesic of the space; the generalized momenta are shown to be the covariant components of the generalized velocity vector. By introducing the time variable explicitly, a space-time metric is formed. The author considers the conditions under which the space part of this metric reduces to that corresponding to classical physics. In this manner, he is able to show that classical mechanics is a first approximation to relativistic mechanics for sufficiently small velocities. The case where the kinetic energy contains the time variable explicitly (as a motion referred to moving axes) is considered and a geometric interpretation by means of geodesics in space-time is given. The action integral, Hamilton's function, the Hamilton-Jacobi equations, the connection between Fermat's principle in geometric optics and Maupertius' principle of least action in mechanics are studied. The case of conservative systems is studied in some detail so that these principles can be fully illustrated. Furthermore, by properly extending the principle of least action, one is shown to be led to one of Boltzmann's theorems in thermodynamics which Ehrenfest has applied to statistical mechanics.

This section closes with an introduction to wave mechanics. The generalization from classical to wave mechanics is made by following de Broglie and assuming that the two theories are related to each other in the same manner as geometrical optics is related to physical optics. Thus, it is shown that if the wave length is sufficiently small (in comparison to the wave amplitude and with little variation in this amplitude) then physical optics (a phenomenon with second order partial differential equation) reduces to geometrical optics (a phenomenon with first order partial differential equation). A reversal of this procedure is to replace the fundamental Hamilton-Jacobi equation of classical dynamics (a first order partial differential equation) by a second order partial differential equation. This furnishes a motivation for the Schrödinger rule. The Schrödinger equation in general coordinates and the commutation rules are discussed by use of the covariant derivative. Finally, groups of waves in space are discussed and it is shown that Planck's quantum relation implies that the classical group velocity of the waves associated with a particle is equal to the particle velocity.

The section on elasticity deals mainly with the theory of the stress and strain tensors, elastic relations and application of this theory to the vibration of crystals. By considering the definition of stress, the author shows that the stress is a pseudo-tensor of the density type. The strain tensor is introduced in the usual manner and some deformation theory is discussed. Further tensors and invariants (related to the internal energy of the body and potential energy of the force system) are studied and the equations of motion in the Eulerian and Lagrangian (due to Boussinesq) coordinates are obtained. The latter equations are applied to the study of general vibrations in crystals, reflections of waves, stationary waves, waves in fluid, pressure of radiation, etc. This section contains, also, a considerable amount of material from the author's papers and earlier books.

In the final section the quantum theory of solids is discussed. Here the author treats the work of Born and Debye.

N. Coburn (Ann Arbor, Mich.).

Coutrez, Raymond. Sur les dérivées covariantes spinorielles et les identités de la physique mathématique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 151–165 (1945).

Let $x^1, \cdots, x^r$ and $\xi^1, \cdots, \xi^p$ be the parametric coordinates for two manifolds $V(x)$ and $\Phi(\xi)$. A connection between V and Φ is introduced by means of the equations $dx^\alpha = h_\beta{}^\alpha(x, \xi)d\xi^\beta$, $d\xi^\beta = h_\alpha{}^\beta(x, \xi)dx^\alpha$. The laws of transformation of tensors with indices belonging to both manifolds are studied in the expected way and are generalized to give the laws of transformation of "generalized spinors." The classical spinors are special cases of these generalized spinors. Covariant generalized spinor derivatives with respect to the indices of either V or Φ are then worked out and are used in a study of the first variation of functions of generalized spinors. The necessary and sufficient conditions that these functions are multipliers with respect to x^α transformations constitute a set of identities with whose help a symmetric impulse-energy tensor is introduced. The paper closes by writing the field equations in generalized Hamiltonian and Jacobian forms. *A. Schwartz.*

ASTRONOMY

Egerváry, E. On a new form of the differential equations of the problem of three bodies. Hungarica Acta Math. 1, 1–18 (1946).

It has been known since Lagrange that the problem of three bodies can be reduced from its original order 18 to a system of order 6 by using the ten elementary integrals, by eliminating the angle of the nodes and by elimination of the time. The explicit writing out of the reduced equations is, however, so complicated that it is rarely carried out. The author obtains a comparatively simple explicit reduction to a system of order 9. Of the six variables which occur in this system, three occur to the first order and three to the second order. The first three are the three components of the angular velocity of the reference frame formed by the three principal axes of inertia of the configuration consisting of the three bodies. The second three are three spatial coordinates to specify the position of each body relative to the axes of inertia. These coordinates are defined symmetrically with respect to the three bodies, after a suggestion of R. Radau [C. R. Acad. Sci. Paris 68, 1465–1468 (1869)]. The possibility of further reduction from order 9 to 6 is at once evident from the obvious presence of two first integrals and from the fact that the time is not explicitly involved. The author uses his reduction for a brief discussion of homothetic solutions. He also considers some particular solutions in the case of equal masses and of forces proportional to the cube of the mutual distances. [It should also be remarked that another explicit reduction to order 8, for some purposes equally simple, has been obtained by A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton University Press, 1941, pp. 318–319; these Rev. 3, 215.] *D. C. Lewis* (College Park, Md.).

***Lévy, Jacques. Les approches dans le problème des trois corps.** Thesis, University of Paris, 1943. 58 pp.

An "approche" in the problem of three bodies is a motion of the three bodies in which one of the mutual distances d meets the conditions $\lim\inf_{t\to\infty} d=0$, $\lim\sup_{t\to\infty} d>0$. The motions are divided into two classes: those in which one mutual distance becomes infinite with t (mouvements à allure binaire) and those for which the mutual distances remain finite or oscillate between finite values and infinity (mouvements bornés et oscillants). The existence of motions with "approche" in the two classes is demonstrated, the methods of proof being analytic for the first class and topological for the second. The thesis also contains a detailed investigation of binary collisions. *M. H. Martin.*

Crenna, Mario. Esposizione elementare di un caso notevole del problema ristretto dei tre corpi. Period. Mat. (4) 23, 28–33 (1943).

Cernet, Paul. Ondes imaginaires dans l'espace à canaux. Suivi de: L'atome et la nébuleuse spirale dans une métrique riemannienne à potentiels dépendant du temps. Ann. Fac. Sci. Univ. Toulouse (4) 7, 1–70 (1 plate) (1945). [MF 16176]

The first part of the paper contains the theory of "canalised" spaces. By this is meant that, by a suitable partitioning of space, certain integrals over surface elements can be propagated by variable surfaces and even by discontinuous fragments of surface, each one propagating itself along its own compartment. Mathematically, the basis is a generalization of Stokes's theorem. Application is made to the differential equation of planetary orbits deduced from Schwarzschild's solution of Einstein's gravitational equations, particularly to the degenerate spiral orbits; also to the propagation of light-waves in classical mechanics but with the assumption that the velocity of light is $c/(1-C/r)$, where c, C are constants and r a polar coordinate. This is formally the velocity of light in the Schwarzschild field.

The second part of the paper deals with space-times whose metrics are of the form $ds^2 = e^\nu dt^2 - e^\mu dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$, where ν, μ are functions of r, t. A roundabout method of analysis due to Delsarte ("binary ds^2") is used to calculate the Einstein gravitational equations with the cosmical constant equal to zero. A solution is then found corresponding to a mass-particle at $r=0$ and an "expanding" distribution of matter-energy outside it of zero density and constant pressure. The paths of light-rays and of planetary particles in this field are worked out in detail. In particular, the degenerate orbits are said to reproduce the arms of spiral nebulae if it is assumed that there are two mass-particles at $r=0$, the fields of which are superposable, one of which has a positive and the other a negative mass. The degenerate orbits are, in certain other cases, regarded as corresponding to the electron system in an atom. *G. C. McVittie.*

Lal, Brij Basi. On the theory of a spiral nebula. Proc. Nat. Acad. Sci. India. Sect. A. 12, 108–120 (1942).

The configuration considered is that of a rotating spheroidal central mass of finite dimensions and of uniform density surrounded by a spheroidal structure of rotating compressible gas of variable density. It is then stated that the equation of the orbit of a particle ejected in the equatorial plane of this system is, in the usual notation,

$$d^2u/d\theta^2 + u = h^{-2}(A_0 + A_1u^2 + A_2u^4 + \cdots + A_nu^{2n}),$$

where $A_0, A_1, \cdots$ are constants. These constants are expressed as functions of other constants ρ_0, k_0, e and a whose significance is not given. If terms involving A_2, A_3, etc. are neglected the orbits, in certain cases, are shown to be "deformed" parabolas and hyperbolas. When the term in

A_2 is included, the equation of the orbit is obtained but no discussion of its shape is attempted. The paper concludes with an attempt to deal with the case when a small frictional resistance proportional to v/r^2 (where v is the velocity of the particle) is introduced. The condition that $dr/d\theta$ should be real, finite and continuous and of the same sign, as θ increases, is found and is said to hold for a spiral orbit, though no proof of the existence of such an orbit is forthcoming [see the four following reviews].
G. C. McVittie (London).

Lal, Brij Basi. The arms of a spiral nebula in resisting medium. I. Proc. Nat. Acad. Sci. India. Sect. A. 13, 19–27 (1943).

The work of the paper reviewed above is continued, the constants A_0, A_1, etc. now being defined in terms of the constants specifying the central mass and the surrounding distribution of rotating compressible gas. The ejected particle is subject to the small resistance v/r^2 as well as to the central force and the equation of the orbit is worked out neglecting all the A_n except A_0 and A_1. This orbit is shown to be, for large radial distances r, of the spiral form $\theta = ar^{-1}+b$, where a and b are constants so that the distance r decreases as θ increases. *G. C. McVittie* (London).

Lal, Brij Basi. On the theory of a spiral nebula. II. Proc. Nat. Acad. Sci. India. Sect. A. 13, 28–36 (1943).

To the central force described in the two papers reviewed above there is now added a small repelling force proportional to the distance from the centre of the distribution. An approximate solution of the orbit is worked out and proves to be, for large values of r, the spiral $r = k\theta + l$, where k and l are constants. *G. C. McVittie* (London).

Lal, Brij Basi. Formation of the arms of a spiral nebula. Proc. Nat. Acad. Sci. India. Sect. A. 13, 179–183 (1943).

The orbit of a particle moving in the gravitational field of a thin disc of matter in such a way that its angular velocity decreases slowly with increasing distance from the centre of the disc is worked out approximately. At great distances from the centre of the disc this orbit is of spiral form. *G. C. McVittie* (London).

Lal, Brij Basi. The arms of a spiral nebula in resisting medium. II. Proc. Nat. Acad. Sci. India. Sect. A. 13, 165–170 (1943).

If, in the investigation contained in part I [see the third preceding review], the resisting force v/r^2 is replaced by a small force proportional to the velocity, the equation of the orbit can again be worked out and again proves to be, for large r, the curve $\theta = ar^{-1}+b$, though with different values of the constants a and b. *G. C. McVittie* (London).

MECHANICS

Dobrovolsky, V. V. Burmester's points in spherical motion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 489–491 (1945). (Russian. English summary) [MF 15442]

In analogy with the motion of a plane in itself the motion of a spherical surface in itself is considered and is shown to lead to analogous concepts, for example, to curves analogous to those of Alt and, in addition to those mentioned in the title, to points analogous to points of Ball. The method is analytic; only Russian literature is quoted.
G. Y. Rainich (Ann Arbor, Mich.).

Ghermanescu, M. Sur les généralisations de la formule de Binet. Mathematica, Timişoara 22, 155–158 (1946).

For a general plane motion, a tangential resistance force is introduced such that the remaining component of force is central. The expression for radial acceleration, except for a variable in place of $h = r^2 d\theta/dt$, is similar to that for central forces. *P. Franklin* (Cambridge, Mass.).

Goldberg, Michael. A three-dimensional analog of a plane Kempe linkage. J. Math. Phys. Mass. Inst. Tech. 25, 96–110 (1946).

It was shown by Kempe [Proc. London Math. Soc. (1) 9, 133–147 (1878)] that, given any plane quadrilateral linkage, one may, without restricting the deformability, join a fifth point by four links to properly selected points on the four sides. The corresponding result for a prismatic linkage is derived at once by using hinges perpendicular to the plane of the quadrilateral, and for a pyramidal linkage by using hinges whose lines are concurrent. A more interesting generalization arises when the four hinges are skew, so that the four bars form the Bennett linkage described in the author's earlier publication [Trans. Amer. Soc. Mech. Eng. 65, 649–661 (1943); these Rev. 6, 74]. Two opposite links of the Bennett linkage have equal "twists," say $2(A+B)$, while the other two links have equal twists $2(C+D)$. If $\tan A \tan B = \tan C \tan D$, then points dividing the first two links in the ratio $\sin 2A : \sin 2B$, and points dividing the other two in the ratio $\sin 2C : \sin 2D$, may be joined by four additional links to a common hinge, without restricting the movability of the linkage. Finally, this theorem is generalized to a Bennett linkage with bent links, so that the points of attachment of the additional links are no longer collinear with the ends of the original links. The paper is illustrated with seven diagrams and seven photographs of working models. *H. S. M. Coxeter* (Notre Dame, Ind.).

Wintergerst, S. Dreh- und Biegeschwingungen in Kardanwellen. Forschung Gebiete Ingenieurwesens. Ausg. B. 13, 213–217 (1942).

In a Cardan linkage (a Hooke's joint, frequently called a universal joint) the relation between the input rotation ϕ and the output rotation ψ is given by $\tan \psi \cos \alpha = \tan \phi$, where the angle α is the deviation from alignment of the shafts. If a uniform rotary motion or a constant turning moment is introduced, the output is a motion or a moment resolvable into a superposition of the input motion and a mixture of frequencies whose fundamental frequency is twice the shaft frequency. The author investigates the torsional oscillations and bending oscillations produced in the driven shaft by this motion. In particular, the two extreme cases are considered. One extreme case occurs when the elastic resistance of the shaft is small compared to the mass attached to it and the other when the elastic resistance is large. Also, the effects of the reaction of the driven shaft upon the driving shaft are considered. *M. Goldberg*.

Bruevič, N. G. The present state and problems of the theory of the precision of mechanisms. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1946, 1065–1079 (1946). (Russian)

Bloh, Z. Š. On the development of approximate methods of synthesis of plane mechanisms. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 683–696 (1946). (Russian)

If $x=\phi_1(t, a_i)$, $y=\phi_2(t, a_i)$ is the trajectory of a point M of a linkage of parameters a_i and if $F(x, y)=0$ is a trajectory to be approximated for $\alpha\leqq t\leqq\beta$, then, if $\theta(t)=F(\phi_1(t, a_i), \phi_2(t, a_i))$ happens to be a polynomial, the linkage parameters a_i furnishing the approximation can be determined from the condition that $\theta(t)$ is a Chebyshev polynomial for the interval $\alpha\leqq t\leqq\beta$. Similar treatment can be given in the case when the trajectory of M is given as $f(x, y)=0$. Three sets of examples are discussed in detail. They constitute the only results of the paper, since the author does not discuss, or even mention, the condition under which a small $\Delta\theta=\theta-\theta_0$ (where the subscript 0 refers to the exact trajectory desired and $\theta_0=0$) entails the smallness of $x-x_0$ and $y-y_0$. This need not be the case when $(F_x')_0$, $(F_y')_0$ happen to be small. *A. W. Wundheiler.*

Cotton, Émile. Sur quelques liaisons imposées à un corps solide. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 101–107 (1946).

In some cases of one solid's sliding on another, to set up the analytic conditions of constraint one must take account of the impossibility of one solid's penetrating another, as well as the conditions of contact. The author solves this problem by considering the velocity distribution in a virtual motion. *P. Franklin* (Cambridge, Mass.).

Jessen, Børge. On the parallelogram of forces. Mat. Tidsskr. A. **1945**, 37–47 (1945). (Danish)

de Possel, René. Les principes mathématiques de la mécanique classique. Gaz. Mat., Lisboa **7**, no. 28, 1–9; no. 29, 9–12 (1946).

Dans ce travail qui s'inspire des travaux de M. Brelot [Ann. Univ. Grenoble **19**, 43–66 (1943); **20**, 1–37 (1944); Les Principes Mathématiques de la Mécanique Classique, Arthaud, Grenoble, 1945; ces Rev. **5**, 16, 190; **7**, 223] M. de Possel montre combien de précision on peut apporter aux fondements mathématiques de la mécanique si l'on regarde les grandeurs fondamentales telles que la masse, la force, l'énergie, etc. comme des fonctions additives d'ensemble et on leur applique systématiquement la notion d'intégrale de Stieltjes. Moyennant la notion générale de torseur réparti, que l'on trouve dans le chapitre III, l'auteur énonce le principe fondamental de la dynamique pour un répère galiléen ou non et une forme généralisée du principe de d'Alembert d'où l'on tire les équations de Lagrange et les transformations de Lagrange et d'Appell. Le chapitre IV, qui porte le titre "Essai d'interprétation physique," est la partie la plus intéressante du travail et la plus difficile à saisir. L'auteur même nous dit que son essai présente encore des difficultés d'interprétation. Il est souhaitable qu'il revienne sur ces questions, en particulier celle de bâtir la mécanique rationnelle sur la seule notion d'énergie. *G. Lampariello* (Messina).

de Possel, René. Sur la définition d'un torseur réparti et sur l'évaluation de sa puissance. C. R. Acad. Sci. Paris **222**, 1470–1472 (1946).

When the external forces on part of a body are equivalent to couples alone, some of the integrals used by Brelot [Les Principes Mathématiques de la Mécanique Classique, Arthaud, Grenoble, 1945; these Rev. **7**, 223] are inapplicable. The author generalizes Brelot's definitions so as to cover this case. *P. Franklin* (Cambridge, Mass.).

de Possel, René. Sur les applications des torseurs répartis à la dynamique des corps à une dimension rectifiables et des milieux continus. C. R. Acad. Sci. Paris **223**, 127–129 (1946).

de Possel, René. Sur l'indétermination de la puissance d'un torseur réparti. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 109–115 (1946).

Proofs of some formulas of use in the mechanics of continuous media, where the differential elements are subjected to couples as well as forces per unit volume. *P. Franklin* (Cambridge, Mass.).

Nielsen, K. L., and Synge, J. L. On the motion of a spinning shell. Quart. Appl. Math. **4**, 201–226 (1946).

The authors tackle from first principles the problem of the motion in air of a spinning shell. Admitting that an exact treatment as a hydrodynamical problem "is obviously out of the question," they follow Fowler, Gallop, Lock and Richmond [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 295–387 (1920)] and Fowler and Lock [ibid. **222**, 227–247 (1921)] in handling the question as an aerodynamical one, namely as one in which "the forces exerted on the shell by the air must be regarded as dependent only on the instantaneous motion of the shell." Two basic ideas are stressed: first, an axis of geometric and dynamic symmetry for the shell; second, the choice of a fixed base point on the axis of the shell, with respect to which the instantaneous force system is mathematically analyzable into a force and a couple, or something equivalent. To insure invariance with respect to shift of mass center, and thus to avoid paradoxes here shown to result from the method of treatment in the work mentioned above, the reference point is regarded as geometrically identified by the surface configuration (say as the centroid), without dependence upon the (symmetric) mass distribution. The aerodynamic functions (expressible in terms of ten positive dimensionless aerodynamic functions) are accepted as presumably adequately determinable from photographs of the shell in flight. The chief properties assumed for the resisting medium reduce to the mathematical ones that it defines (i) a scalar field of air density, (ii) a scalar field of local sound velocity, (iii) a vector field of velocity. This last defines two other vector fields, vorticity and acceleration. The aerodynamic force system exerted on the solid by the fluid consists of an aerostatic force and an aerokinetic force system. At one stage, linearity of four of the functions in terms of four velocity and angular velocity components is assumed and some of the angles are assumed to be small.

Fourteen fundamental quantities are defined, necessarily departing somewhat from classical terminology because of the changed reference system adopted for the resolution. The names proposed are (1) cross velocity, (2) axial velocity, (3) cross spin, (4) axial spin, (5) cross force due to cross velocity, (6) Magnus force due to cross velocity, (7) Magnus force due to cross spin, (8) cross force due to cross spin, (9) axial force, (10) Magnus torque due to cross velocity, (11) cross torque due to cross velocity, (12) cross torque due to cross spin, (13) Magnus torque due to cross spin, (14) Magnus axial torque. (For uniformity the viscous axial torque is also called "Magnus.") In terms of these the differential equations are set up and studied.

To effect a practical solution of the differential equations, consistent with data that would be provided by photographs, a partial linearization of the equations is made by neglecting second order terms in quantities known to be small for acceptable projectiles, namely the rate of turning of the vertical plane through the axis of the shell and the angle of yaw.

New light is cast upon the problem of stability. Employing usual assumptions as to physical data, three inequalities are obtained, only the first of which has been current in the literature. Again, drift of the projectile seems here adequately accounted for even if Magnus effects are negligible.

A. A. Bennett (Providence, R. I.).

Bottema, O. The stability of Staude's top-motion. Nederl. Akad. Wetensch., Proc. **48**, 316–325 (1945). (Dutch) [MF 16586]

Staude has shown [J. Reine Angew. Math. **113**, 318–334 (1894)] that the motion under gravity of every rigid body one of whose points O is fixed can be a uniform rotation around each one of a system of ∞^1 axes through O. This rotation axis has to be vertical. The stability of the Staude rotations has been investigated by Grammel [Math. Z. **6**, 124–142 (1920)] who considered the point O, the axis of rotation l and the principal moments of inertia at this point as given. Then he determined the region R in which the center of gravity must lie in order that the uniform rotation around l be stable. The author considers the more direct question: given a rigid body with a fixed point O, which axes of Staude are stable? This problem is completely solved by the author in the case in which the center of gravity lies on one of the principal axes of inertia through O.

J. Haantjes (Amsterdam).

De Simoni, Franco. Sul moto dei corpi rigidi con sospensione quasi-baricentrica. Ann. Scuola Norm. Super. Pisa (2) **11**, 197–210 (1942). [MF 16760]

If a rigid body is free to rotate about a fixed point whose distance r from the center of gravity is small enough so that r^n ($n\geqq 2$) can be neglected, we have so-called quasi-barycentric suspension. The author's considerations are for the most part limited to the case when the moment of momentum is (neglecting an infinitesimal in r) parallel to a constant force applied at the center of gravity.

D. C. Lewis.

Mangeron, D. Sur le mouvement de certains systèmes articulés de corps rigides pesants. C. R. Acad. Sci. Paris **223**, 190–191 (1946).

The author gives the differential equations for the general multiple pendulum together with the linear equations for small oscillations. Deductions from these equations are reserved for future notes.

D. C. Lewis.

Zahradníček, Josef. Étude des oscillations non amorties d'un système de pendules de torsion couplés. Publ. Fac. Sci. Univ. Masaryk **1946**, no. 277, 18 pp. (1946). (Czech. French summary)

Dans le présent travail je donne, en appliquant les équations d'énergie, la solution du problème de k ($=1, 2, 3, 4$) pendules de torsion non amortis couplés les uns aux autres. Les pendules sont supposés suspendus dans la même verticale à un fil élastique, l'extrémité inférieure du fil étant aussi fixée. Les fréquences des oscillations couplées sont calculées dans le cas général et dans des cas spéciaux, où l'on suppose que les moments d'inertie sont égaux, ainsi que les élasticités, de sorte que les fréquences des oscillations propres des différents pendules sont les mêmes.

From the author's summary.

Zahradníček, Josef. Ungedämpfte Schwingungen zweier gekoppelter Torsionspendel. Časopis Pěst. Mat. Fys. **70**, 133–152 (1941). (Czech. German summary)

Kazinsky, V. A. The influence of difference in amplitude and phase on the periods of two conjugate pendulums. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 577–578 (1946).

Tatevsky, V. Vibrations of polyatomic molecules. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **15**, 445–458 (1945). (Russian. English summary) [MF 15380]

Having in view actual computations in concrete cases the author begins with a discussion of a general mechanical system in which the potential and kinetic energies are quadratic forms in the generalized coordinates and velocities respectively; making extensive use of matrices, he introduces several forms of the equations involving also generalized forces and impulses. One of his systems is a general form of a system introduced in a special case by Eliashevich; he shows that certain quantities appearing in it are Maxwell's coefficients of mobility. Simple physical interpretations of the individual elements of the matrices mentioned above are given and made use of in two examples, that of a linear molecule and that of a molecule consisting of three atoms not in a line. The results are those of Eliashevich, but the method is less laborious, according to the author. Using transformations of coordinates he investigates the structure of a system in terms of several kinds of dependences among coordinates, such as geometrical dependence and kinematical dependence.

G. Y. Rainich.

Hostinský, Bohumil. Sur la répartition d'énergie dans les spectres acoustiques. C. R. Acad. Sci. Paris **222**, 1328–1329 (1946). [MF 16738]

A system of n particles restricted to translations has a finite number of normal frequencies referred to as the acoustical spectrum. Assume n very large. Let $e(\nu)$, $f(\nu)d\nu$ and $dE(\nu)$ be the energy for frequency ν and the number of normal frequencies and energy, respectively, in the range ν to $\nu+d\nu$. Then $dE(\nu)/d\nu=f(\nu)e(\nu)$. The author gives two examples of possible densities $f(\nu)$ and remarks that for the Maxwell-Boltzmann distribution $e(\nu)$ is constant, but $f(\nu)$ is not; that is, the energy is not distributed with density independent of frequency.

D. G. Bourgin (Urbana, Ill.).

Fuertes, Fidel Alsina. On the "power function." J. Appl. Phys. **17**, 712 (1946).

An explicit expression is given for the power function introduced by D. A. Wells [J. Appl. Phys. **16**, 535–538 (1945); these Rev. **7**, 90] in case the system consists of a number of particles acted upon by forces directed along the tangents of the trajectories. The force on the kth particle may depend arbitrarily on the velocity of that particle, on the coordinates of all the particles and on the time.

D. C. Lewis (College Park, Md.).

Andronov, A., et Neumark, G. Sur les mouvements des modèles idéalisés d'horloge ayant deux degrés de liberté. Horloge pré-galiléenne. C. R. (Doklady) Acad. Sci. URSS (N.S.) **50**, 17–20 (1946).

The nonconservative nonlinear dynamical system afforded by the elements of an idealized clock, which react with each

other through a series of imperfectly elastic collisions, leads to the consideration of a transformation. The fixed point of this transformation represents the periodic motion of the system and the condition for this periodic motion to be stable is easily obtained. The authors remark that the relationship of this problem to more orthodox problems in nonconservative dynamics is analogous to the relationship of the Birkhoff billiard ball problem to problems regarding conservative systems. *D. C. Lewis* (College Park, Md.).

García, Godofredo. Most general cardinal and scalar equations of the dynamics of bodies with internal movement. Actas Acad. Ci. Lima **9**, 43–110 (1946). (Spanish)

García, Godofredo. On the most general form of the equations of dynamics for holonomic and nonholonomic systems. Actas Acad. Ci. Lima **9**, 119–136 (1946). (Spanish)

These papers are concerned with the description of the motion of a dynamical system in terms of several moving reference frames. *W. Kaplan* (Ann Arbor, Mich.).

Thomas, T. Y. On the transformation of the equations of dynamics. J. Math. Phys. Mass. Inst. Tech. **25**, 191–208 (1946).

The author considers two Lagrangian systems D and E of more than two degrees of freedom with the property that a one-to-one correspondence exists between the coordinates of these systems by means of which all trajectories of D will be mapped into trajectories of E and conversely. For two corresponding coordinate systems (x^α) the differential equations for the trajectories are

$$\text{(D)} \qquad \frac{d^2x^\lambda}{dt^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=Q^\lambda,$$

$$\text{(E)} \qquad \frac{d^2x^\lambda}{d\tau^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=R^\lambda,$$

where Q^λ and R^λ are the nonvanishing generalized force vectors, functions of x alone, and the Γ's and Λ's are Christoffel symbols derived from the coefficients of the expressions for the kinetic energies of the two systems

$$\tfrac{1}{2}g_{\mu\lambda}\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt}, \quad \tfrac{1}{2}h_{\mu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau}.$$

Necessary and sufficient conditions in order that all trajectories with real parameter t of D will be trajectories with real τ of E and conversely are that

$$\Lambda^\lambda_{\mu\nu}-\Gamma^\lambda_{\mu\nu}=\delta_\mu{}^\lambda\varphi_\nu+\delta_\nu{}^\lambda\varphi_\mu; \quad R^\lambda=c^2(g/h)^{2/(n+1)}Q^\lambda=c^2e^{-4\varphi}Q^\lambda,$$

where $\varphi_\nu=\partial_\nu\varphi$, c is a constant and $g=\det(g_{\mu\lambda})$, $h=\det(h_{\mu\lambda})$. In addition, the author gives in case of conservative fields the general form of $h_{\mu\lambda}$ and the potential function W (of system E) in terms of $g_{\mu\lambda}$ and the potential function V (of D) under the assumption that all the trajectories of E are trajectories of D. *J. Haantjes* (Amsterdam).

Thomas, T. Y. The transformation of dynamical systems of two degrees of freedom. Proc. Nat. Acad. Sci. U. S. A. **32**, 106–111 (1946). [MF 16363]

Let D and E be dynamical systems with n degrees of freedom, which are referred to the same set of generalized coordinates x^i. The Lagrange equations are

$$\text{(D)} \qquad \frac{d^2x^h}{dt^2}+\Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=Q^h(x),$$

$$\text{(E)} \qquad \frac{d^2x^h}{d\tau^2}+\Lambda^h_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=R^h(x),$$

where Γ and Λ are Christoffel symbols derived from the coefficients g_{ij} and h_{ij} of the expressions for the kinetic energies of D and E, respectively. It is assumed that the force vectors Q and R do not vanish in the region considered. In the paper reviewed above, necessary and sufficient conditions in order that all the trajectories of E are trajectories of D have been obtained for $n>2$. In this paper it is shown that all results obtained for $n>2$ are valid for systems with two degrees of freedom. Conditions are given for the identity of all trajectories of D and E in the general case and for conservative systems. One of the necessary conditions is that h_{ij} has to be a linear expression in the coefficients $g^{(\alpha)}_{ij}$ of a basis of quadratic first integrals of the differential equations of the geodesics of D. *J. Haantjes.*

Forbat, N. Sur la séparation des variables dans l'équation de Hamilton-Jacobi d'un système non conservatif. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 462–473 (1946).

The well-known results of Levi-Civita, Dall'Acqua and Stäckel on separable dynamical systems are modified so as to include cases where the Hamiltonian depends on t.
D. C. Lewis (College Park, Md.).

Nagabhushanam, K. The action form and Jacobi's last multiplier. Proc. Nat. Acad. Sci. India. Sect. A. **11**, 69–72 (1941).

Let the Pfaffian form of action $(p_\lambda dq^\lambda-Hdt)$ be written as $X_i dx^i$, where the x^i stand for the $2n+1$ variables p_λ, q^λ and t. Then the equations of motion of a conservative holonomic dynamical system can be written as $a_{ij}dx^j=0$, where $a_{ij}=\partial_iX_j-\partial_jX_i$. The cofactors A^{ij} of a_{ij} in the skew symmetric matrix a_{ij} have the weight two and can be written as $A^{ij}=\alpha^i\alpha^j$, the matrix a_{ij} being of odd order. It is shown that $\operatorname{div}\alpha^i=0$ and that $\alpha^i\epsilon_{i_1\cdots i_{2n}i}$ is a Stokes tensor; $\epsilon_{i_1\cdots i_{2n+1}}$ is the $(2n+1)$-vector density. It follows that

$$\int\alpha^iX_idx^1\cdots dx^{2n+1}, \quad \int\alpha^i\epsilon_{i_1\cdots i_{2n}i}dx^{i_1}\cdots dx^{i_{2n}}$$

are invariant integrals connected with the last multiplier of Jacobi for the differential equations of the trajectories.
J. Haantjes (Amsterdam).

Aymerich, Giuseppe. Trasformazione non esattamente adiabatica ed integrazione approssimata di un particolare sistema multiperiodico nel caso degenere. I. Rend. Sem. Fac. Sci. Univ. Cagliari **14**, 1–13 (1944). [MF 16224]

L'auteur, ayant étudié dans un travail antérieur l'intégration approchée d'un système d'équations hamiltoniennes avec n degrés de liberté à solutions approximativement périodiques, dans l'hypothèse des périodes differentes pour chaque variable lagrangienne, envisage ici le cas dégénéré dans lequel un certain nombre de périodes coincident. Il donne quelques formules préliminaires pour n quelconque; à la suite il applique ces formules pour calculer, dans le cas de $n=2$, une valeur approchée de l'erreur. La solution proposée semble au rapporteur beaucoup compliquée pour le but de l'approximation. *B. Levi* (Rosario).

Aymerich, Giuseppe. Trasformazione non esattamente adiabatica ed integrazione approssimata di un particolare sistema multiperiodico nel caso degenere. II. Rend. Sem. Fac. Sci. Univ. Cagliari 15, 14–26 (1945). [MF 16630]

Résumé de l'auteur: Généralisation formelle des résultats de la note I [voir l'analyse ci-dessus] au cas d'un nombre quelconque de degrés de liberté et d'une dégénération d'ordre m; le calcul dépend de la résolution d'un système canonique d'ordre $2m$, qu'on intègre dans quelques cas particuliers.

B. Levi (Rosario).

Sofrin, Thomas G. The combination of dynamical systems. J. Aeronaut. Sci. 13, 281–288 (1946). [MF 16782]

A method using tensor algebra is presented to demonstrate the forced vibration characteristics of a system formed by coupling a group of subsystems. The results depend upon the coupling constants. *A. E. Heins.*

Hydrodynamics, Aerodynamics, Acoustics

Costa de Beauregard, Olivier. Sur la conservation de la masse propre. Sur la notion de fluide parfait. C. R. Acad. Sci. Paris 222, 271–273 (1946). [MF 16000]

The paper deals with the (special) relativistic motion of a fluid under the action of a body 4-force which is not in general perpendicular to the 4-velocity. *J. L. Synge.*

Costa de Beauregard, Olivier. Équations générales de l'hydrodynamique des fluides parfaits. C. R. Acad. Sci. Paris 222, 369–371 (1946). [MF 16015]

The paper deals with a relativistic fluid. The 4-velocity V^i is related to an expansion σ and a skew-symmetric vorticity tensor τ^{ij} by the equations $\partial_i V^i=\sigma$, $\partial^j V^i-\partial^i V^j=\tau^{ij}$. The problem is to solve these equations when the fields σ and τ^{ij} are given, the latter satisfying $\sum\partial^i\tau^{jk}=0$, where $\sum$ indicates summation over a cyclic permutation. A solution is given by writing $V^k=\partial^k P+\partial_l R^{kl}$, where P and R^{kl} satisfy $\partial_i{}^i P=\sigma$, $\partial_i{}^i R^{kl}=\tau^{kl}$, $\sum\partial^i R^{jk}=0$. With this type of solution, we have $\partial_i{}^i V^k=\partial^k\sigma+\partial_l\tau^{kl}$. The above equations involving the operator $\partial_i{}^i$ admit solutions of the retarded type. Solutions in terms of a vector potential A^i, such that $\partial_j{}^j A^i=V^i$, are also considered. *J. L. Synge* (Pittsburgh, Pa.).

Costa de Beauregard, Olivier. Sur la théorie des forces élastiques. C. R. Acad. Sci. Paris 222, 477–479 (1946). [MF 16027]

The paper contains some applications of the Green-Stokes theorems to integrals involving pressure and more general stresses in a relativistic fluid. *J. L. Synge.*

Kriesis, P. Über die Symmetrie des Spannungstensors in der Aerodynamik. Z. Physik 122, 778–786 (1944).

The author gives two examples of discontinuous velocity fields in which the stress tensor is finite but asymmetric.

C. C. Torrance (Annapolis, Md.).

Kiveliovitch, Michel. Sur l'équation de Boltzmann. C. R. Acad. Sci. Paris 222, 1273–1275 (1946). [MF 16733]

Soit f la solution de l'équation intégro-différentielle de Boltzmann, f_0 la fonction de Maxwell, solution de cette équation dans le cas stationnaire. L'auteur cherche f sous la forme $f=(f_0/\lambda)(1+\lambda\phi_1+\lambda^2\phi_2+\cdots)$, où λ désigne un paramètre. Chaque approximation doit vérifier cinq conditions. La première, commune à toutes les approximations, se trouve être la classique équation de continuité des fluides compressibles. Les quatre autres varient d'une approximation à l'autre et l'auteur les explicite pour les trois premières. Pour les première et seconde on retombe sur diverses équations de la dynamique des fluides. Pour la troisième, la forme des conditions à satisfaire est beaucoup plus compliquée. *J. Kravtchenko* (Grenoble).

Moisil, Gr. C. Sur le passage des variables de Lagrange aux variables d'Euler en hydrodynamyque. Bull. Math. Soc. Roumaine 44, 55–58 (1942). [MF 12749]

The author points out that the fact that the velocity of a particle of a fluid is independent of the initial time is nontrivial. *A. Gelbart* (Syracuse, N. Y.).

Ballabh, Ram. On fluid motions superposable on an irrotational motion. Proc. Benares Math. Soc. (N.S.) 7, 11–15 (1945).

The author solves the equation $(\nabla\times q)\times Q=\nabla\chi$ for the irrotational velocity Q for the special case where the components of q are $u=az-by$, $v=bx-cz$, $w=cy-ax$, where a, b, c are constants and χ is a function of one, two or three space variables. *C. C. Torrance* (Annapolis, Md.).

Cârstoiu, J. Sur la condition des accélérations dans un fluide incompressible. Mathematica, Timişoara 20, 172–173 (1944).

Kaufmann, Walther. Die kinetische Energie der von einem Wirbelpaar erzeugten Flüssigkeitsbewegung. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1943, 295–306 (1944).

The kinetic energy of the two-dimensional velocity field produced by a vortex pair is calculated by a direct integration under the assumption that the vortex cores have essentially a rigid body rotation. The result agrees closely with the value predicted by Prandtl [Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. 1919, 107–137]. *G. F. Carrier.*

Doucet, E. Hydraulique théorique et appliquée. Étude de quelques solutions des équations de l'hydrodynamique. Fluides non visqueux et fluides visqueux. Mouvements permanents. Assoc. Franç. Avancement Sci. Séances de Sections 63 (1939), 41–58 (1941). [MF 16909]

The author obtains particular solutions of the equations of steady state motion of a viscous or nonviscous fluid by specializing the character of the solution. Using the decomposition of the velocity vector V in the form $V=\operatorname{grad}\theta+U$, where $\operatorname{div} U=0$, he obtains several families of solutions explicitly for each of the following three specializations: (1) U is a function of a scalar $\phi(x, y, z)$; (2) $\phi=\theta$; (3) $\operatorname{rot} U=U$. The behavior of the solutions as the viscosity vanishes is observed and several cases are shown where the solution does not reduce to that for the case of zero viscosity. The solutions are not made to fit boundary conditions of any kind. *D. Gilbarg* (Bloomington, Ind.).

Mukherjee, Santi Ram. Motion of an incompressible fluid with varying coefficient of viscosity, given by $\mu=\mu_0+\epsilon_1 x$, for positive values of x. I. Proc. Nat. Acad. Sci. India. Sect. A. 12, 121–130 (1942).

A fluid as defined in the title is considered, x being one of the rectangular space coordinates. After omitting certain terms from the equations of motion pertaining to this fluid, steady state solutions are investigated. The one particular solution obtained shows certain of the omissions to be invalid. *G. F. Carrier* (Providence, R. I.).

Mukherjee, Santi Ram. Motion of an incompressible fluid with varying coefficient of viscosity, given by $\mu=\mu_0+\epsilon_1 x$, for positive values of x. II. Proc. Nat. Acad. Sci. India. Sect. A. 12, 151–160 (1942).

The paper reviewed above is extended to include phenomena which decay exponentially with the time. The omissions are still inconsistent. *G. F. Carrier.*

Mukherjee, Santi Ram. Motion of an incompressible fluid with varying coefficient of viscosity, given by $\mu=\mu_0+\epsilon_1 x$, for positive values of x. III. Proc. Nat. Acad. Sci. India. Sect. A. 12, 161–176 (1942).

The work of the papers reviewed above is continued. This time the necessary terms are retained. Steady state and exponentially time decaying solutions are considered. The solutions are obtained, using separation of variables, in the form of products of Bessel functions and unwieldy power series. *G. F. Carrier* (Providence, R. I.).

Mukherjee, Santi Ram. Motion in incompressible fluid of variable density. Proc. Nat. Acad. Sci. India. Sect. A. 13, 1–18 (1943).

This paper considers the motion of a fluid with a density which depends on the coordinate x. The equations of motion are linearized by writing the velocity as $\bar{v}=U+u_1, u_2, u_3$ and neglecting higher-order terms in u_i/U. The results are again in the unwieldy form of the papers reviewed above. *G. F. Carrier* (Providence, R. I.).

Mukherjee, Santi Ram. Motion in incompressible fluid of variable density. II. Proc. Nat. Acad. Sci. India. Sect. A. 13, 135–150 (1943).

The paper reviewed above is extended to include the exponentially decaying time dependent solutions. *G. F. Carrier* (Providence, R. I.).

Dolidze, D. E. On the existence of solutions of the non-linear problem of hydrodynamics. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 11–16 (1943). (Russian. Georgian summary) [MF 11714]

In previous papers [same Bull. 1, 659–664 (1940); 2, 499–506 (1941); these Rev. 3, 219; 5, 247] the author reduced the integration of the equation of motion of a viscous incompressible fluid (Navier-Stokes equations and the continuity equation) in a three-dimensional domain under suitable initial and boundary conditions to the solution of a system of nonlinear integro-differential equations. He showed that a solution exists for all values of the time t, provided the Reynolds number is sufficiently small. In this note another sufficient condition is derived which will be satisfied for all values of the Reynolds number provided that t does not exceed a certain limiting value. This value depends upon the domain considered, the density of the fluid and the prescribed values of the speed at the boundary. The method of proof consists in solving the system by successive approximations and determining sufficient conditions for convergence. *L. Bers* (Syracuse, N. Y.).

Dolidze, D. On the hydrodynamical stream function. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 623–631 (1943). (Georgian. Russian and English summaries) [MF 11715]

In previous papers Leray [J. Math. Pures Appl. (9) 12, 1–82 (1933)] and the author [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 7, 65–103 (1940); same Bull. 3, 649–656 (1942); these Rev. 3, 218; 6, 24] proved the existence of a solution of the equation $\nu\Delta\Delta\psi-\partial\Delta\psi/\partial t=f(x,y,t)$, satisfying suitable initial and boundary conditions. The equation is the "linearized" equation of the stream-function of an incompressible viscous flow in two dimensions. In this note an integral representation of the solution is given, involving a suitably defined "Green's function" $G(P,Q,t)$ [concerning the definition of G, see the following review]. *L. Bers* (Syracuse, N. Y.).

Dolidze, D. On certain properties of the hydrodynamical Green's function. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 373–382 (1944). (Georgian and Russian) [MF 14614]

In the paper reviewed above the author defined a Green's function for the equation

$$\Delta(\nu\Delta\psi-\psi_t)=f(x,y,t) \tag{1}$$

which is satisfied by the stream-function of a two-dimensional viscous flow. This function $G(P,Q,t)$ may be interpreted as the value at P of the stream-function of a flow in the domain considered which is due to a concentrated mass force acting at the point Q. In this paper he proves the symmetry of G as well as the fact that G may be represented as a limit of appropriately chosen regular solutions of (1). *L. Bers* (Syracuse, N. Y.).

Dolidze, D. Solution of Prandtl's equation of the non-stationary boundary layer. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 867–876 (1944). (Georgian. Russian summary) [MF 14615]

According to the Russian summary, the author proves the existence of a solution of the initial and boundary value problem for the differential equation of an unsteady boundary layer. This is done by transforming the problem into a system of three nonlinear integro-differential equations for the functions u (tangential velocity), u_x, u_y. This system is solved by a successive approximation method. The solution obtained is valid for sufficiently small values of the time variable t. *L. Bers* (Syracuse, N. Y.).

Jaeger, Charles. Note sur une forme très générale de l'équation des courants liquides permanents à surface libre. C. R. Acad. Sci. Paris 223, 417–419 (1946).

The author records the equations pertaining to a steady liquid stream with a free surface. *G. F. Carrier.*

Dumitrescu, D. T. Quelle bezw. Senke im Kreisrohr. Bull. École Polytech. Bucarest [Bul. Politechn. București] 15, 70–76 (1 plate) (1944). [MF 13582]

The author deals with the problem of a flow through a circular channel with a distribution of sources and sinks on the symmetric axis. In order to increase the rapidity of the convergence of the series for the potential and stream functions, as given by Lamb, spherical functions are used. By beginning with one and then adding more and more singularity points to the symmetric axis the author hopes to approximate given closed boundaries, though no attempt is made to consider conditions of convergence of the method. The cases of one source and two sources are carried out in detail giving graphs and tables of values for the stream and potential functions. *A. Gelbart* (Syracuse, N. Y.).

Vedernikov, V. V. On the calculation of the unsteady flow of a fluid in an open channel. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1946, 499–504 (1946). (Russian)

Efros, D. A. Hydrodynamical theory of two-dimensional flow with cavitation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 267–270 (1946).

Gurevič, M. I. Symmetrical flow with cavitation over a flat plate placed between parallel walls. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 487–498 (1946). (Russian)

Čarnyĭ, I. A. Approximate calculation of drag for cavitational separation of flow of an incompressible fluid around a solid body. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 935–941 (1946). (Russian)

Oudart, A. Sur le schéma de Helmholtz-Kirchhoff. J. Math. Pures Appl. (9) **22**, 245–320 (1943). [MF 12251]
Oudart, A. Sur le schéma de Helmholtz-Kirchhoff. J. Math. Pures Appl. (9) **23**, 1–36 (1944). [MF 13795]

The author investigates the problem of the existence of a wake produced by a given obstacle in a curvilinear channel. The first part deals with a preliminary analysis of the inverse (or indeterminate) problem as given by H. Villat [Aperçus theoriques sur la resistances des fluides, Collection Scientia, no. 38, Gauthier-Villars, Paris, 1920]. It is shown that the traditionally accepted limitation of the velocity, as postulated by M. Brillouin [Ann. Chimie Physique (8) **23**, 145–230 (1911)] cannot hold for a certain class of obstacles and channels. The maximum of the velocity is then attained on the obstacle and not at infinity and the free stream lines are not necessarily convex with respect to the domain of the fluid in motion. In the second part, several existence theorems are established, generalizing J. Kravtchenko's results for rectilinear channels [J. Math. Pures Appl. (9) **20**, 35–234, 235–303 (1941); these Rev. **3**, 219; **4**, 58]. The investigation is based on previous papers by A. Weinstein, C. Jacob and J. Leray. *A. Weinstein* (Pittsburgh, Pa.).

Yadoff, Oleg. Sur les écoulements à la Poiseuille. C. R. Acad. Sci. Paris **223**, 192–193 (1946).

L'auteur étudie le problème de l'écoulement à la Poiseuille d'un fluide pesant, incompressible, visqueux, à travers un tube cylindrique à section droite polygonale. Dans le cas du rectangle, il obtient la loi de répartition des vitesses le long des axes de symétrie de la section sous forme de séries très rapidement convergentes, qu'il explicite dans le cas du carré et qui se prêtent mieux au calcul numérique que les formules données autrefois par Boussinesq [J. Math. Pures Appl. (2) **13**, 377–424 (1868), particulièrement pp. 385–389]. *J. Kravtchenko* (Grenoble).

Meksyn, D. Fluid motion between parallel planes. Dynamical stability. Proc. Roy. Soc. London. Ser. A. **186**, 391–409 (1946).

The author studies the stability of laminar motion between parallel planes by using the equation of Sommerfeld, which has been studied by Heisenberg, Tollmien and others. He claims that "the results, however, of Heisenberg and Tollmien are invalid, since their transformations of the nonviscous integrals through the critical point are incorrect." He reconsiders the problem discussing, however, only the case of neutral disturbances and using only asymptotic expressions to specify a solution. [Reviewer's remark: as an asymptotic expansion does not specify a function uniquely, the author's argument does not seem to be conclusive.]

The author reaches the following conclusions. (1) The critical region is the source where vibrations are generated. (2) Curved profiles admit a periodic motion at sufficiently high Reynolds numbers. (3) In particular, parabolic flow is unstable at high Reynolds numbers with a critical number of 6700, based on the velocity at the center of the channel and its half-width. The corresponding wave length of the disturbance is about three times the width of the channel.

[Reviewer's remarks. The conclusion (2) has been known for some time for (a) symmetrical profiles and (b) profiles of the boundary-layer type. For these cases, simple formulae for the calculation of the critical Reynolds number are also known [Lin, Proc. Nat. Acad. Sci. U. S. A. **30**, 316–323 (1944); Quart. Appl. Math. **3**, 117–142, 218–234 (1945); 277–301 (1946); these Rev. **6**, 190; **7**, 225, 226, 346]. None of these papers are quoted by the author. It seems that the author's discussion is limited to case (a). The result (3) agrees completely with the earlier results contained in the papers quoted above, to within the accuracy of the approximations. These earlier calculations were based on the work of Heisenberg, with only slight improvements. It would be rather paradoxical if either the work of Heisenberg or that of the present author were essentially wrong. A physical picture more concrete than (1) has been suggested before.] *C. C. Lin* (Providence, R. I.).

Kiebel, I. A. A case of unhomogeneous turbulence in a compressible fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 244–247 (1945).

The author considers turbulent flow of a heavy compressible fluid. Instead of pressure p and density ρ the variables θ and π are used which are defined as $\theta = p^{1/k}/\rho$, $\pi = (k/(k-1))p^{(k-1)/k}$, $k = c_p/c_v$. The variables u, v, w, θ and π are assumed of the type $\varphi(x, y, z, t) = \bar{\varphi}(z) + \varphi'(x, y, z, t)$, where $\bar{\varphi}$ denotes the mean motion, φ' the fluctuating motion. Following Friedmann and Keller, correlation coefficients $B_\varphi{}^\psi$ are introduced, where

$$B_\varphi{}^\psi = \overline{\varphi'(x-\xi, y-\eta, z-\zeta, t-\tau)\psi'(x+\xi, y+\eta, z+\zeta, t+\tau)};$$

φ' and ψ' stand for any of the fluctuating quantities. Neglecting triple correlations, a system of equations involving the $B_\varphi{}^\psi$ is obtained in the standard fashion from the equations of motion. The number of possible correlation coefficients is then reduced by assumptions concerning symmetry and homogeneity. An explicit solution can then be given in terms of Bessel functions.

The author proposes to apply the solution to the problem of turbulent flow of air above a nonuniformly heated plate and also to the problem of the diffusion of a shock wave. *H. W. Liepmann* (Pasadena, Calif.).

Velikanov, M. A. Kinematical structure of turbulent flow in open channels. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **10**, 331–340 (1946). (Russian. English summary)

Nevzgljadov, V. A phenomenological theory of turbulence. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 614–625 (1946). (Russian. English summary)

An English translation appeared in Acad. Sci. USSR. J. Phys. **9**, 235–243 (1945); these Rev. **7**, 347.

Teletov, S. G. Sur le mouvement lent séparé des mélanges gaz-liquide. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 179–182 (1946).

Shchelkachev, V. N. Fundamental equations of motion of compressible fluids through compressible media. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 103–106 (1946).

Shchelkachev, V. N. Analysis of unidimensional motion of a compressible fluid in a compressible porous medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 203–206 (1946).

Ferri, Antonio. Application of the method of characteristics to supersonic rotational flow. Tech. Notes Nat. Adv. Comm. Aeronaut. no. **1135**, 25 pp. (15 plates) (1946).

Dalitz, R. H. Some mathematical aspects of compressible flow. Austral. Counc. Aeronaut. Rep. **ACA-20**, 39 pp. (1946).

An expository report.

Krivoshein, Nicolás. Conditions of applicability of the equation of Daniel Bernoulli. Revista Unión Mat. Argentina **11**, 184–205 (1946). (Spanish. English summary)

The conditions of applicability of Daniel Bernoulli's equation to the movement of a perfect compressible fluid are discussed. *From the author's summary.*

Ville, Jean. Sur l'équation de la force vive dans le mouvement rectiligne des gaz. C. R. Acad. Sci. Paris **223**, 308–310 (1946).

The energy equation applying to a one-dimensional gas flow is juggled into a somewhat unconventional form. For the case where a discontinuity exists a certain assumption is made concerning this discontinuity (that is, shock wave). The assumption leads to a relation which contradicts the known result that stagnation enthalpy is conserved across a shock [cf. H. W. Emmons, J. Aeronaut. Sci. **12**, 188–194, 216 (1945); these Rev. **6**, 191] and thus must be incorrect.
G. F. Carrier (Providence, R. I.).

Jacob, Caius. Sur une interprétation de l'équation de continuité hydrodynamique. Bull. Math. Soc. Roumaine Sci. **46**, 81–90 (1944). [MF 16511]

D. Pompeiu proved [Rend. Circ. Mat. Palermo **54**, 113–123 (1930)] that in two dimensional steady flow of an incompressible fluid the area A bounded by a closed curve C, the area A_H bounded by the hodograph of the velocity vectors for the points of C and the area A_V bounded by the loci of the extremities of velocity vectors for the points of C are related by $A_V=A+A_H$. This theorem is generalized to steady and nonsteady compressible two dimensional flows but it is shown that a direct generalization to three dimensional flows is not possible. *M. H. Martin.*

Jacob, Caius. Sur l'emploi de la méthode hodographique en mécanique des fluides compressibles. Mathematica, Timişoara **22**, 170–181 (1946).

It is well known that the stream function ψ (or the potential φ) of a two-dimensional compressible flow satisfies a linear differential equation in the independent variables V and θ (magnitude and direction of the vector velocity). After demonstrating this result, the author derives a new equation, which serves to define $\omega=\theta+i\log V$ as a function of $f=\varphi+i\psi$ and its conjugate $\bar{f}$. Setting

$$\omega=\omega_0+\omega_1M^2+\omega_2M^4+\cdots,$$

he shows how it is possible to compute the successive terms to any order of approximation. The new method is thus superior to Chaplygin's, which breaks down after the second approximation because of the use of an artificial equation of state.

As in any application of the hodograph method it is in a practical sense impossible to satisfy given boundary conditions. These must in general be derived a posteriori.
D. P. Ling (Murray Hill, N. J.).

Daymond, S. D. Further remarks on the two-dimensional motion of a compressible fluid. Quart. J. Math., Oxford Ser. **17**, 129–130 (1946).

The author gives a derivation of W. Tollmien's spiral flow [Z. Angew. Math. Mech. **17**, 117–136 (1937)] by his hodograph method. *H. S. Tsien* (Cambridge, Mass.).

Servanty, Lucien. Sur une forme de solution générale de l'équation hodographique de l'écoulement plan d'un fluide compressible, utilisant les fonctions analytiques. C. R. Acad. Sci. Paris **221**, 283–284 (1945). [MF 14500]

The hodograph equation of a plane flow of a compressible fluid is

$$\frac{W^2(1-W^2)}{1-(\gamma+1)/(\gamma-1)W^2}\frac{\partial^2\chi}{\partial W^2}+\frac{\partial^2\chi}{\partial\theta^2}+W\frac{\partial\chi}{\partial W}=0,$$

where W is a dimensionless local velocity, θ the direction of the velocity and γ the usual ratio C_p/C_v. The author considers series solutions of the form $\chi=\sum_0^\infty W^{2n}Z_n$, where the Z_n are analytic functions of the complex variable $z=We^{i\theta}$. The Z_n are given by the recursion formula

$$Z_{n+1}=\frac{n(2\gamma-1)+2}{2(\gamma-1)(n+1)}Z_n-\frac{zZ_n'}{2(\gamma-1)(n+1)}-\frac{n^2+n(2\gamma-1)+2}{2(\gamma-1)(n+2)z^{n+1}}\int_{z_0}^{z}Z_nz^ndz;$$

Z_0 is an arbitrary function. Under restricted conditions some results are obtained concerning the singularities of the solutions and the domain of convergence. *A. Gelbart.*

Sedov, L. I. Unsteady motions of compressible fluids. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 293–311 (1945). (Russian. English summary) [MF 15340]

Employing the theory of dimensionality, the author determines a number of exact solutions of the equations of one-dimensional steady motion of a compressible fluid in the case of plane waves and motions with cylindrical or spherical symmetry. He determines solutions which depend upon constants among which either two or only one have independent dimension. These solutions make it possible to construct the solutions for the problem of explosions along a plane, the problem of detonation, the problem of a gas under constant pressure, the problem of the motion of a gas towards and away from a center. The paper gives a complete solution of these problems in the case of spherical symmetry. For the general case, the solution is reduced to the integration of a differential equation of the first order. In some particular cases, the author obtains the solutions in finite form, containing an arbitrary constant.
S. Bergman (Cambridge, Mass.).

Emmons, Howard W. The theoretical flow of a frictionless, adiabatic, perfect gas inside of a two-dimensional hyperbolic nozzle. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1003**, 24 pp. (34 plates) (1946).

Birkhoff, Garrett. Reversibility and two-dimensional airfoil theory. Amer. J. Math. **68**, 247–256 (1946). [MF 16421]

Definition. A theory of fluid dynamics will be called incomplete if its conditions do not determine the steady flow around a body uniquely; incorrect if its predictions do not agree closely with experimental fact. Theorem 1 (reversibility paradox). Any reversible theory of fluid dynamics is either incomplete or grossly incorrect, so far as its predictions of steady state lift and drag are concerned. Proof. Such a theory will predict that a steady flow and its reverse will give the same pressure thrust on an obstacle, whereas it is a matter of common experience that a flow and its reverse ordinarily give pressure thrusts in approximately opposite directions. Theorem 2 (restricted d'Alembert paradox). Any complete reversible theory of fluid dynamics must predict zero lift and drag for steady flow about a body symmetric in a point or in a plane perpendicular to the line of flow. Proof. We assume tacitly that the theory is invariant under rigid transformations of space, hence that reflection in a point or in a plane of symmetry replaces each flow permitted by the theory. However, with a steady flow, the same effect can be achieved by reversing time (writing $-t$ for t; changing the sign of the velocity vector). Hence, if the theory is complete, the two must be identical, and the pressure distribution must have the symmetry described. The conclusion is now obvious.

The author then goes on to state that two-dimensional aerofoil theory must "fail grossly" when the angle of attack exceeds the stalling angle and that the consideration of "pathological shapes" yields absurd conclusions. On the other hand, he fails to point out that classical hydrodynamics can draw correct inferences only as to the behaviour of a hypothetical ideal fluid and that, from its very nature, it can yield no criterion for the applicability or otherwise of these conclusions to an actual fluid.

L. M. Milne-Thomson (Greenwich).

Theodorsen, Theodore. Note on the theorems of Bjerknes and Crocco. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1073**, 4 pp. (1946).

The theorems of Bjerknes and Crocco are of great interest in the theory of flow around airfoils at Mach numbers near and above unity. A brief note shows how both theorems are developed by short vector transformations.

Author's summary.

Bers, Lipman. Velocity distribution on wing sections of arbitrary shape in compressible potential flow. I. Symmetric flows obeying the simplified density-speed relation. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1006**, 32 pp. (2 plates) (1946).

This is the first step in a program to calculate the velocity distribution of arbitrary airfoils, with circulation, in subsonic flow of a compressible fluid. In this paper the profiles considered are symmetrical, in symmetrical flow without circulation, and the density-speed relationship of Tchaplygin, von Kármán, and Tsien is used.

The transformation mapping the region exterior to the airfoil profile in the z-plane onto the domain in a σ-plane exterior to a slit, where σ is the complex potential, is sought; it is not conformal, but is conformal with respect to a metric involving the fluid velocity. [See also L. Bers, same Tech. Notes, no. **969** (1945); these Rev. **7**, 497.] The domain exterior to the slit in the σ-plane is mapped conformally on a ζ-plane, exterior to a circle. Then, in particular, a function $f(\omega)$ is sought, relating the points of the airfoil contour in the z-plane to the points of the circle $\zeta=Re^{i\omega}$. It is found that the surface velocity at the airfoil can be calculated in terms of $f(\omega)$ in two different ways. By combining these formulas, a nonlinear integral equation for $f(\omega)$ is obtained. This is solved by a method of iteration. It is shown that the velocity distribution at points not on the airfoil contour is theoretically determined by knowledge of $f(\omega)$. Finally, two numerical examples are given: (i) a circular cylinder at a stream Mach number of 0.406 and (ii) a Joukowski profile at 0.685. In these cases the velocity distributions are computed to the fifth and fourth approximations, respectively, and are compared with the distributions in incompressible flow and those according to von Kármán and Tsien [von Kármán, J. Aeronaut. Sci. **8**, 337–356 (1941); these Rev. **3**, 220] and Kaplan [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **621** (1938)]. *W. R. Sears* (Ithaca, N. Y.).

Bers, Lipman. Velocity distribution on wing sections of arbitrary shape in compressible potential flow. II. Subsonic symmetric adiabatic flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1012**, 53 pp. (4 plates) (1946).

The methods used here are similar to those of the note reviewed above, but are generalized to handle any given pressure-density relationship, provided the flow remains everywhere subsonic. In this case the transformation from the airfoil (z) plane to the circle (ζ) plane is made via the hodograph plane and a "distorted hodograph" plane, that is, the w^*-plane, where w^* is related to the complex velocity in the z-plane is a manner involving the speed. Again the transformation $\zeta(z)$ is conformal with respect to a metric involving the speed. The function $f(\omega)$ is again required, since it determines the velocity field. A relation between the surface values of $|w^*|$ and $f(\omega)$ is determined by a function-theoretical argument that makes use of the boundary condition at the surface. This leads ultimately to a functional equation satisfied by $f(\omega)$; this is proved to be equivalent to the boundary-value problem.

To solve the functional equation an iteration procedure is used. The convergence of the procedure is not proved, but is expected to be satisfactory, on the basis of experience with the case represented in the note reviewed above. The procedure involves calculation of various functions after an approximation to $f(\omega)$ is chosen; one of these steps requires the numerical integration of a linear partial differential equation with variable coefficients; it is proposed that computing-machine methods be employed here. Finally, this theory is compared with other methods, including that of the first paper, to which it reduces in the case of the approximate velocity-density relation. A velocity correction formula proposed by Garrick and Kaplan [Nat. Adv. Comm. Aeronaut., Advance Confidential Rep. no. **L4C24** (1944)] is discussed. *W. R. Sears* (Ithaca, N. Y.).

Kaplan, Carl. Effect of compressibility at high subsonic velocities on the lifting force acting on an elliptic cylinder. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1118**, 29 pp. (1 plate) (1946).

The author expands the two-dimensional stream function in a series each term of which is a function of the physical

coordinates, the angle of attack (assumed small) and the thickness ratio. Each of the perturbation terms is assumed of the first order of smallness with respect to the preceding in the thickness ratio, with a similar remark applying to the derivatives. This form is better adapted than a straight perturbation series to the examination of bodies having a stagnation point. Substituting this expansion in the basic differential equation for the stream function, and equating terms of equal orders of magnitude, the author obtains the differential equations for the first two disturbance terms of the series. The first represents the Prandtl-Glauert linear approximation, the second the next higher correction term.

An affine transformation, depending on the free stream Mach number, is used to reduce the Prandtl-Glauert equation to the potential equation. Since the affine transform of an ellipse is another ellipse, separate calculations are not required for every Mach number, which instead is carried through simply as a parameter. By the use of conformal mapping, the first two disturbance terms are found for a general elliptic cylinder at a small angle of attack. The exact boundary conditions are satisfied, as well as the Kutta condition of no flow at the end of the major axis. The ratio of lifts for compressible and incompressible fluids is given as a simple expression in the thickness ratio and Mach number and the results are displayed tabularly and graphically. *D. P. Ling* (Murray Hill, N. J.).

Sears, W. R. On compressible flow about bodies of revolution. Quart. Appl. Math. **4**, 191–193 (1946). [MF 16964]

The author definitely settles the question of whether there is a correction factor for the disturbance velocity and for the pressure coefficient of a slender body of revolution placed in a uniform stream of compressible fluid. It is shown that, although for infinitesimal slenderness ratio there is no correction, for the small but finite slenderness ratio concerned in engineering practice there is a correction factor. This factor is approximately $(1-M^2)^{-\frac{1}{2}}$, where M is the Mach number of the free stream. *H. S. Tsien.*

Lees, Lester. A discussion of the application of the Prandtl-Glauert method to subsonic compressible flow over a slender body of revolution. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1127**, 17 pp. (1 plate) (1946).

Previous writers on this subject have obtained erroneous results by applying the boundary condition (of tangential flow at the body) at an approximate radius, after making the Prandtl-Glauert affine transformation. This process, which is valid in the analogous two-dimensional case, leads to considerable errors here. That this must be so is evident from the fact that the radial velocity component, for any source-sink distribution along the axis, varies rapidly with radius at small radii, while the axial component does not. The author satisfies the (linearized) boundary condition at the correct radius and shows that a universal velocity-correction formula applicable to all shapes is impossible in the three-dimensional case. He calculates the correction for the special case of ellipsoids of revolution.

The bibliography includes a list of papers in which the theory has been applied incorrectly [the reviewer's note reviewed above is not mentioned, but belongs in the same category]. Correct calculations have previously been made by Göthert [Lilienthal Gesellschaft für Luftfahrtforschung, Bericht **127**, 97–101 (1940)] and Bilharz and Hölder [Deutsche Luftfartforschung, Forschungsbericht **1169/1**]. *W. R. Sears* (Ithaca, N. Y.).

Lighthill, M. J. The supersonic theory of wings of finite span. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2001** (**8079**), 15 pp. (1944).

A fundamental solution of the linearized potential equation for supersonic irrotational flow of a compressible fluid was used by von Kármán and Moore [Trans. A. S. M. E. **54**, 303–310 (1932)]. It represents, within the Mach cone opening downstream from the origin, the potential due to a source in a supersonic stream. By integrating the effects of a distribution of such sources in a plane, the author is able to calculate the potential due to a symmetrical wing of finite dimensions, at zero incidence. By similar treatment of the elementary solution used by Tsien in the case of bodies of revolution at angles of incidence [J. Aeronaut. Sci. **5**, 480–483 (1938)] he finds the potential for a flat-plate wing at angle of incidence. This treatment follows that of Schlichting [Luftfahrtforschung **13**, 320–335 (1936)], but certain errors have been corrected. The author calculates the lift, drag, and moment coefficient. *W. R. Sears* (Ithaca, N. Y.).

Lighthill, M. J. Supersonic flow past bodies of revolution. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2003** (**8321**), 24 pp. (1945).

The author employs the solutions used by von Kármán [Atti del V Convegno della Fondazione Alessandro Volta, Rome, 1935, pp. 222–276] and von Kármán and Moore [Trans. A.S.M.E. **54**, 303–310 (1932)] for symmetrical supersonic flow over slender bodies of revolution, according to the linear perturbation theory, but first investigates in some detail the order of approximation involved. From these solutions he obtains an integral for the wave drag, which is the same as that used by von Kármán [loc. cit.] although no reference is given. The author then seeks the projectile shape for minimum wave drag for given length and caliber, assuming the minimum-drag shape to be symmetrical fore-and-aft. This shape is determined, but is rejected as inadmissible in the present theory because the ends are not actually pointed. The approximate pressure distributions are plotted for a parabolic shape at various Mach numbers, as well as typical streamlines. For projectiles with flat sterns (shells) a similar investigation is made to determine the shape for minimum wave drag (excluding the drag of the blunt stern); the result, of course, is the ogive of von Kármán [loc. cit.] although no reference is mentioned. This shape is also rejected as being actually inadmissible. The theory is next extended to the exterior and interior flows of a hollow projectile or tube.

The final paragraphs concern projectiles at yaw, following the work of Tsien [J. Aeronaut. Sci. **5**, 480–483 (1938)]. It is shown that the lift vanishes if the body is pointed at both ends. Again the accuracy of the approximation is investigated and it is concluded that a statement of Tsien regarding the relative accuracies of two calculations is not correct. *W. R. Sears* (Ithaca, N. Y.).

Jones, Robert T., and Margolis, Kenneth. Flow over a slender body of revolution at supersonic velocities. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1081**, 8 pp. (7 plates) (1946).

The theory of small disturbances is applied to the calculation of the pressure distribution and drag of a closed body of revolution traveling at supersonic speeds. It is shown that toward the rear of the body the shape of the pressure distribution is similar to that for subsonic flow. For fineness

ratios between 10 and 15 the theoretical wave drag is of the same order as probable values of the frictional drag.

Authors' summary.

Jones, Robert T. Thin oblique airfoils at supersonic speed. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1107, 21 pp. (12 plates) (1946).

The well-known methods of thin-airfoil theory have been extended to oblique or swept-back airfoils of finite aspect ratio moving at supersonic speeds. The cases considered thus far are symmetrical airfoils at zero lift having plan forms bounded by straight lines. Because of the conical form of the elementary flow fields the results are comparable in simplicity to the results of the two-dimensional thin-airfoil theory for subsonic speeds. In the case of untapered airfoils swept back behind the Mach cone the pressure distribution at the center section is similar to that given by the Ackeret theory for a straight airfoil. With increasing distance from the center section the distribution approaches the form given by the subsonic-flow theory.

From the author's summary.

Puckett, Allen E. Supersonic wave drag of thin airfoils. J. Aeronaut. Sci. 13, 475–484 (1946).

An elementary solution of the linear differential equation of the small-perturbation theory for supersonic flows is

$$d\phi(x,y,z)=q\,d\xi\,d\eta\{(x-\xi)^2-\beta^2[(y-\eta)^2+z^2]\}^{-\frac{1}{2}},$$

where $\beta^2=M^2-1$ and M is the stream Mach number. This is the potential due to a source distribution of strength q over the area $d\xi d\eta$ at ξ,η; it is shown that the vertical velocity produced by this distribution vanishes for $z=0$ except over the area $d\xi d\eta$, where it has the value $\pm\pi q$. Hence the boundary conditions for any thin symmetrical airfoil (without lift) will be satisfied by a suitable distribution of such sources.

The author treats first a two-dimensional infinite wing swept back at angle σ to the flow, where (a) $\tan\sigma<\beta$ and (b) $\tan\sigma>\beta$. He calculates, by integration over the wing area, the tangential velocity component at the surface and hence the pressure coefficient and the drag (wave drag) coefficient. In both cases the results agree with those obtained from the theory of the infinite swept back wing by superposition of flows [R. T. Jones, Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1033** (1946)]; for example, the drag vanishes in case (b) because the velocity component normal to the leading edge is subsonic. The author then proceeds to consideration of wings having planforms in the shape of an isosceles triangle (apex forward) and having double-wedge profiles, that is, "delta wings." The fundamental problem here is to compute the pressure distribution over a delta in which there is a constant source distribution. The complete wings can then be obtained by superposition of deltas. Three distinct cases arise as the pertinent Mach cones lie behind and ahead of the leading edge and/or the maximum-thickness line. The drag coefficient is calculated for each of these cases and is plotted as function of the geometrical parameters.

For the case in which the leading edge of a delta wing lies ahead of the Mach cone originating at the nose, the author's methods can be used to attack the problem of a flat-plate wing at an angle of incidence. The result obtained is that the lift coefficient is the same as for a two-dimensional (not swept back) supersonic airfoil.

W. R. Sears.

Stewart, H. J. The lift of a delta wing at supersonic speeds. Quart. Appl. Math. 4, 246–254 (1946).

The linear differential equation of the small-perturbation theory for three-dimensional supersonic flow is transformed by introduction of new coordinates r, μ, θ, such that θ is constant on meridional planes through the x-axis (in the direction of the stream), μ is constant on cones about that axis and r is constant on hyperboloids. The resulting equation has solutions of the form

$$\sum_{m,n} A_{mn}\begin{Bmatrix} r^n \\ r^{-n-1}\end{Bmatrix}\begin{Bmatrix} P_n^m(\mu) \\ Q_n^m(\mu)\end{Bmatrix}\begin{Bmatrix}\cos(m\theta) \\ \sin(m\theta)\end{Bmatrix},$$

where P_n^m and Q_n^m are Legendre functions. If P denotes one of the Cartesian velocity components (u,v,w) or the pressure, enthalpy or acceleration potential, the case $n=0$ represents the "conical flows" of Busemann [Schr. Deutsch. Akad. Luftfahrtforschung **7B**, 105–121 (1943); Luftfahrtforschung **12**, 210–220 (1935)]. In such flows the fluid properties are constant along radial lines emanating from the origin. For conical flows, the differential equation can be reduced by a simple transformation to Laplace's equation in two dimensions. The author therefore employs the theory of functions of a complex variable to obtain useful solutions. He is able to determine general differential relationships between the various functions whose real parts are u, v and w.

As an example, he calculates the lift of a delta wing (triangular planform, apex forward) when the leading edge lies within the Mach cone of the apex, at a small angle of attack. By a series of conformal transformations, the region under consideration is mapped in the cells of a plane where the solution is recognized in terms of Jacobian elliptic functions. The final result is a simple expression for the slope of the lift curve. In the limit of a very sharply pointed planform the result agrees with that of R. T. Jones cited in the preceding review. In the other limit, where the leading edge touches the Mach cone, the result agrees with that of Puckett [see the preceding review], that is, the two-dimensional value is obtained. The results are presented graphically.

W. R. Sears (Ithaca, N. Y.).

Couchet, Gérard. Sur les mouvements plans non stationnaires infiniment voisins de mouvements à circulation constante. C. R. Acad. Sci. Paris **222**, 170–171 (1946). [MF 15990]

In a preceding note [same C. R. **221**, 280–282 (1945); these Rev. **7**, 343] the author determined those two-dimensional ideal flows about airfoils in nonuniform motion which are possible with constant circulation. He now considers motions with slightly varying circulation. The reasonable assumption is made that in first approximation the motion of the shed vorticity (trailing surface of discontinuity) which is connected with the varying circulation is determined by the flow with constant circulation. On the basis of this assumption there is obtained an integral equation for the circulation as a function of time and of the motion of the airfoil. This equation contains as a special case the corresponding known result of thin-wing theory for nonuniform motion of a rigid wing section.

E. Reissner.

Serebriisky, J. M. Flow past a symmetrical aerofoil. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 476–478 (1945).

Well-known formulas of the thin-airfoil theory [see, for example, von Kármán and Burgers, Aerodynamic Theory (W. F. Durand, editor), vol. 2, chap. 2, Springer, Berlin,

1935] are obtained by starting with elliptic coordinates, $x=\mathrm{ch}\,\psi\cos\theta$, $y=\mathrm{sh}\,\psi\sin\theta$, and approximating for small values of ψ on the surface of the airfoil. Apparently the author proposes to carry higher-order terms in certain formulas, to avoid troubles near the leading edge, but this does not appear to be done consistently. The analogous formulas for the case of longitudinal flow past a body of revolution are also given, with reference to an earlier paper [same C. R. (N.S.) **41**, 150–153 (1943); these Rev. **6**, 77].

W. R. Sears (Ithaca, N. Y.).

Brennan, M. J., and Stevenson, A. C. Simplified two-dimensional aerofoil theory. A basic method of determining the profiles of laminar flow and high-speed aerofoils. Aircraft Engrg. **18**, 182–186, 194 (1946). [MF 16857]

H. J. Stewart [J. Aeronaut. Sci. **9**, 452–456 (1942); these Rev. **4**, 121] has given a simplified thin-airfoil theory. The authors apply Stewart's method to find the camber and thickness function of an airfoil for a given pressure distribution and also to obtain the hinge moment of a flap.

H. W. Liepmann (Pasadena, Calif.).

Dorodnicyn, A. A. Generalization of the lifting-line theory for cases of a wing with a curved axis and a slipping wing. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 33–64 (1944). (Russian. English summary) [MF 11468]

The immediate application of the Prandtl lifting-line theory to the case of a slipping wing or a wing with a curved axis (swept wing) is impossible because the induced velocities calculated at the lifting-line are infinite. This investigation is based on the following method. The velocities are calculated not at the lifting-line itself, but in the neighborhood of this line, and then represented in form of a series in terms of $\epsilon=r/l$, where r is the distance from the lifting-line in the plane of the section of the wing considered and l is the span. In the general case this series contains, besides a constant term, a logarithmic one, which tends to infinity when $\epsilon\rightarrow 0$. The other terms of the series are of order $\epsilon\log(1/\epsilon)$; hence they are negligible in the neighborhood of the wing for a large aspect-ratio.

The circulation in some section of the wing is determined by these two main terms in the expression for the velocity. The part of the circulation produced by the constant term is determined by the ordinary Prandtl relation. In order to determine the part of the circulation produced by the logarithmic term it is necessary to solve the problem of flow with such a logarithmic singularity near the section of the wing. To the third power of the angle β of roll (or sweep) the logarithmic flow is potential in the plane of the section of the wing; hence the problem in question can be solved by means of the conformal representation of the wing section on a circle.

From the author's summary.

Panichkin, I. A. On the theory of a wing in a flow of circular cross section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 312–317 (1945). (Russian. English summary) [MF 15341]

The author computes the distribution of circulation $\Gamma(x)$ across a wing of finite span enclosed in a circular jet. The function $\Gamma(x)$ satisfies the integro-differential equation

$$\Gamma(x)=A\alpha_0+B\int_{-s}^{s}\Gamma'(t)\{(t-x)^{-1}\pm(R^2-tx)^{-1}\}dt,$$

where A and B are constants depending upon the density, the speed at infinity and the aspect ratio, s is the span, α_0 the angle of attack and R the radius of the jet. The upper (lower) sign holds for an open (closed) jet. The solution is assumed to be of the form $\sum G_n\sin(2n+1)\varphi$, where $\varphi=x/s$, and an infinite system of linear equations is obtained for the coefficients G_n.

A numerical example is given and wind-tunnel correction coefficients are determined for the angle of attack and for the induced drag. It turns out that the values of these coefficients are practically the same as those given by the more primitive theory which assumes an elliptic lift distribution.

L. Bers (Syracuse, N. Y.).

Roy, Maurice. Écoulement permanent subsonique pour une grille de profils. C. R. Acad. Sci. Paris **223**, 496–498 (1946).

Baranov, V. Oscillations d'un disque circulaire plongé dans un liquide visqueux: une solution nouvelle. Cahiers de Physique no. **15**, 43–48 (1943).

The author shows that the resistance of a viscous liquid to a disc plunged into it is proportional to the fractional derivative (of order one-half) of the angular velocity of the disc. The disc is suspended horizontally by a string with a given coefficient of torsion fixed at its center. The result is obtained by solving the differential equations involved by the Laplace transform.

A. E. Heins (Pittsburgh, Pa.).

Cotton, Émile. Essai de théorie des nappes liquides fermées de Savart. Ann. Univ. Lyon. Sect. A. (3) **4**, 75–81 (1941).

Savart [Annales de Chimie et de Physique (2) **54**, 55–87, 113–145 (1833)] has investigated the problem wherein a vertical jet of fluid issues from a horizontal circular orifice and strikes a coaxial circular disc. He predicts that for certain values of the parameters (flow rate, surface tension, etc.) the thin sheet of water leaving the disc will close, that is, that $r(z)$ defines a closed surface of revolution. The author proves the existence of surfaces (sheets of fluid) which have a vertical tangent within a bounded region but is silent on the question of closure. He recommends a numerical procedure for computing the function $r(z)$, as the differential equation is not easily solved analytically, and casts some doubts on the adequacy of Savart's tabulated figures for this purpose.

G. F. Carrier.

Taub, A. H. Interaction of progressive rarefaction waves. Ann. of Math. (2) **47**, 811–828 (1946).

When discontinuities in pressure and velocity are absent, the method of solution for problems of one-dimensional propagation of a disturbance of finite amplitude in a non-viscous, nonheat-conducting gas undergoing a polytropic process was given by Riemann. If the polytropic constant γ is equal to $(2n+1)/(2n-1)$, where n is an integer, the equation reduces to the Euler-Poisson equation and the solution can be explicitly given in terms of two arbitrary functions. The author gives the detailed application of this process to (a) interaction of two expansion waves, (b) reflection of a progressive wave from a rigid wall and (c) reflection of a progressive wave from a density discontinuity.

H. S. Tsien (Cambridge, Mass.).

Dean, W. R. On the reflexion of surface waves by a submerged plane barrier. Proc. Cambridge Philos. Soc. **41**, 231–238 (1945). [MF 13443]

The author considers the influence on waves of small amplitude in deep water of a transverse plane barrier with

bounding edge submerged a distance a below the surface. By familiar procedure, the problem is reduced to that of determining a complex function in the half-plane $y<0$, satisfying appropriate conditions at $y=0$ (the surface, where pressure is zero), and on the half-line $x=0$, $y<a$ (the barrier, where the velocity vector is vertical); the real part of the function is the velocity potential. The required function is found as a sum, of which one component represents the classical standing wave, while the other mirrors the effect of the barrier on the incident progressive wave. From this solution asymptotic expressions for the distorted displacement of the surface at large distances from the barrier are obtained. Numerical data are given on the dependence of the amplitude of the reflected and transmitted waves on the parameter a. *J. W. Calkin* (Houston, Tex.).

Wedernikow, V. V. Conditions at the front of a translation wave disturbing a steady motion of a real fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 239–242 (1945). [MF 16661]

For the case mentioned in the title, this paper confirms by actual computation the general fact concerning characteristics that the partial derivatives can be computed along a characteristic in terms of one partial derivative which remains arbitrary, provided the characteristic relations are satisfied. *D. Gilbarg* (Bloomington, Ind.).

Landau, L. D. Impact waves far from their source. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 286–292 (1945). (Russian. English summary) [MF 15339]

This note deals with shock-waves due to a body which moves with a steady supersonic speed. At large distances from the body the shock-waves may be considered to be sonic disturbances such that the cubes of the amplitudes may be neglected. Introducing a new independent variable $\tau=x(U^2-c_0^2)^{-\frac{1}{2}}$, the author reduces the problem to the equation of spherical waves, τ being the time-coordinate; here x is the coordinate in the direction of the motion of the body, U the speed of the body and c_0 the speed of sound at a stagnation point. A small portion of the wave may be considered as a plane wave whose amplitude decreases as $r^{-\frac{1}{2}}$, r being the distance from the x-axis. The author concludes that there will be two shock waves following the body and that the intensity of the shock is proportional to $r^{-\frac{3}{4}}$. The same method is applied to a brief discussion of spherical shock-waves due to an explosion. The presentation is rather sketchy and the reviewer was unable to follow all details. *L. Bers* (Syracuse, N. Y.).

Levinson, J. I. L'étude de la stabilité des courants supersoniques de gaz en relation avec une double solution de la théorie des ondes de choc. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 151–170 (1945). (Russian. French summary) [MF 14081]

In a steady plane oblique shock the velocity vector $\mathbf{V}_2$ behind the shock is not uniquely determined by the velocity $|\mathbf{V}_1|$ in front of the shock and by the angle $\alpha=\sphericalangle(\mathbf{V}_1, \mathbf{V}_2)$. In fact, for given values of $|\mathbf{V}_1|$ and α the "shock conditions" yield three different values of $|\mathbf{V}_2|$, only of one of which can be excluded on the basis of the second law of thermodynamics. Experimental evidence, however, shows that of the remaining two cases only one is possible, namely, the one corresponding to the larger value of $|\mathbf{V}_2|$ and the smaller angle between $\mathbf{V}_1$ and the shock line. The present paper is an attempt to supply a theoretical explanation of this phenomenon. The author investigates the stability of the well-known "flow against a wedge" using the method of small perturbations in such a way that the perturbations of the shape of the shock line are taken into account. He derives the following necessary condition of stability: the component of $\mathbf{V}_2$ in the direction of the shock line must exceed the speed of sound in the undisturbed flow, that is, the speed of sound corresponding to the local speed $|\mathbf{V}_1|$. Except for very high Mach numbers this condition excludes one of the two values of $|\mathbf{V}_2|$ mentioned above.

Using the stability condition the author computes the smallest possible value $M_{\min}(\alpha)$ of the Mach number for which a steady rectilinear shock is possible. This value is higher than the one previously indicated by von Kármán. In particular, the author obtains $M_{\min}=\sqrt{2}$ for $\alpha=0$, whereas according to von Kármán $M_{\min}=1$. *L. Bers* (Syracuse, N. Y.).

Sretenskiĭ, L. N. On the waves produced by a ship moving in a circular path. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 13–22 (1 plate) (1946). (Russian)

Udeschini, Paolo. Propagazione delle onde finite di marea. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **78**, 320–332 (1943). [MF 16246]

Taking only the hypotheses (a) that the fluid is frictionless and incompressible, (b) that the vertical component of the acceleration of the fluid is negligible compared with the acceleration due to gravity, the author applies Levi-Civita's method of characteristic varieties [Caratteristiche dei Sistemi Differenziali e Propagazione Ondosa, Bologna, 1931] to show that the speed of propagation in a canal or in an unbounded ocean of depth h is $(h+y)\sqrt{g}$, where y is the elevation of the free surface above the mean level. *L. M. Milne-Thomson* (Greenwich).

Ratser-Ivanova, F. S. Semi-diurnal tides in a two-dimensional infinite channel of constant depth rotating with constant angular velocity. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **10**, 373–382 (1946). (Russian. English summary)

The equation for semi-diurnal tides in two-dimensional basins is solved with the aid of conformal mapping of the contour confining the basin upon the upper half-plane. The problem is solved for the case of an infinite channel of constant depth rotating with constant angular velocity. The solution is given for a special form of the boundary condition. The following suppositions on tidal waves are made: we consider long waves and neglect vertical accelerations; the amplitude of the oscillations of the liquid's particles is regarded as small compared with the depth of the channel. The solution is obtained as a series by the small parameter method. *Author's summary.*

Van Mieghem, Jacques. Forme intrinsèque du critère d'instabilité dynamique de E. Kleinschmidt. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 19–33 (1945).

The air is said to be in "dynamic equilibrium" in a given region if the wind is geostrophic. Let c_p be the specific heat at constant pressure, R the gas constant, both for air, p the pressure, $p_0=1000$ mb, $\pi=(c_p/A)(p/p_0)^{AR/c_p}$, T the temperature, θ the potential temperature, $\boldsymbol{\omega}$ the vector of the earth's rotation, $\mathbf{u}$ the geostrophic wind. Then the equilib-

rium condition may be written in the form $2\omega\cdot\nabla u=\nabla\theta\times\nabla\pi$. In order to investigate the nature of the dynamic equilibrium, an infinitely small virtual displacement of a particle from its equilibrium position is considered. It is shown that the dynamic equilibrium is stable or unstable depending on whether the quadratic form $a_{yy}\Delta y^2+2a_{yz}\Delta_y\Delta_z+a_{zz}\Delta z^2$ (Δy and Δz, components of the virtual displacement) is positive or negative. The sign of this form depends on the sign of $a_{yz}^2-a_{yy}a_{zz}=2\omega\cdot\nabla\pi(\nabla\theta\cdot\text{curl }\mathbf{U})$ where $\mathbf{U}=\mathbf{u}+\omega\times\mathbf{r}$, the air motion in an absolute coordinate system. Finally, the relation of Kleinschmidt's criterion to the equation for the frequency of inertia motion obtained by Solberg is shown.
B. Haurwitz (Cambridge, Mass.).

Van Mieghem, Jacques. Relation d'identité entre la stabilité de l'équilibre dynamique de E. Kleinschmidt et la stabilité des oscillations d'inertie de l'atmosphère terrestre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 134–143 (1945).

In the paper reviewed above, the close relation was pointed out between the criterion for dynamic stability and the frequency of the inertia oscillations. In the present paper it is shown that two types of inertia oscillations are possible in the atmosphere, one with a period of about 10 minutes, the other with a period of 15 to 20 hours. The shorter of these is connected with the hydrostatic stability, the longer with the dynamic stability of the atmosphere, and the corresponding short- and long-period oscillations are stable or unstable depending on whether the condition for static or dynamic equilibrium is satisfied or not. *B. Haurwitz.*

Van Mieghem, Jacques. Contribution à la dynamique des surfaces de discontinuité de l'atmosphère. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 481–493 (1946).

For the study of the dynamics of atmospheric surfaces of discontinuity it is convenient to use a right-hand coordinate system whose x-axis is parallel to the intersection of the surface of discontinuity with the horizontal, whose y-axis is parallel to the surface of discontinuity and in the direction of strongest ascent of the surface and whose z-axis is perpendicular to the surface of discontinuity and upwards. Let s denote the specific volume, p the pressure and let brackets indicate the difference between the values of the same variable on both sides of the discontinuity. Furthermore, the z-component of the vector of the earth's rotation is $\omega_z=\omega(-\cos\phi\cos\beta\sin\alpha+\sin\phi\cos\alpha)$, where ω is the angular velocity of the earth's rotation, ϕ the latitude, α the inclination of the surface of discontinuity, β the angle between the x-axis and the east direction. The surface of discontinuity is then a surface of upsliding [Aufgleitfläche] or descending motion [Abgleitfläche] depending on whether $[v_x]$ is less than or greater than $-([s^*]/2\omega_z)\partial p/\partial y$.
B. Haurwitz (Cambridge, Mass.).

Van Mieghem, Jacques. De la convergence additionnelle et du mouvement vertical de l'air dans les dépressions d'altitude associées aux ondes du front polaire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 61–78 (1946).

In the middle tropopause a pressure wave is assumed which is superimposed on the general pressure gradient from north to south. The amplitude of this wave decreases northward and southward, upward and downward, and may also be a function of the time. The velocity components for this pressure field are determined from the author's simplified equations of motion. If the pressure amplitude is independent of time, divergence and convergence occur because of the difference between the air speeds in troughs and ridges. The troughs are then lines of divergence, the ridges lines of convergence. If the pressure amplitude increases (decreases) with time, additional convergence (divergence) occurs in the troughs, and additional divergence (convergence) in the ridges. Furthermore, a ridge is strengthened (weakened) with ascending (descending) motion. In a trough the opposite rule holds. *B. Haurwitz* (Cambridge, Mass.).

Van Mieghem, Jacques. Quelques formes des bilans énergétiques des fluides parfaits en mouvement relatif lorsque le mouvement d'entrainement est une rotation. Assoc. Franç. Avancement Sci. Séances de Sections **63** (1939), 28–33 (1941). [MF 16908]

It is shown that the equations for the kinetic energy as well as the equations resulting from a combination of the first law of thermodynamics and the hydrodynamic equations of motion are the same in an absolute and a rotating coordinate system only if the angular velocity of the rotating coordinate system is independent of the time. Such is the case on the rotating earth. The amount of heat received per unit mass and time is equal to the change of internal energy per unit mass and time plus the work done by the pressure on the unit mass per unit time even if the system rotates with a variable angular velocity. *B. Haurwitz.*

Cagniard, Louis. Étude théorique des surfaces de discontinuité du second ordre dans l'atmosphère. Cahiers de Physique no. 6, 10–18 (1941).

The author develops a theory of surfaces of discontinuity of the second order. These are, in the author's terminology, surfaces where the wind velocity and the temperature are continuous while their first derivatives with respect to the space coordinates change abruptly. Formulas are given which show how the inclination of these surfaces of discontinuity depends on the discontinuities of the vertical gradients of wind velocity and temperature. These formulas apply, in particular, to the tropopause, which is such a surface of discontinuity of the second order. The derivation of similar formulas in "Physikalische Hydrodynamik" by V. Bjerknes and others [Springer, Berlin, 1933] is shown to be unsatisfactory, although the final result is practically identical with that obtained by the author. Finally, a relation is derived between height variations of the tropopause and the discontinuities of the wind and temperature gradients. The available observations are not accurate enough for a rigorous check of these expressions, but they indicate at least that the order of magnitude of the height variations comes out correctly. *B. Haurwitz* (Cambridge, Mass.).

Neamtan, S. M. The motion of harmonic waves in the atmosphere. J. Meteorol. **3**, 53–56 (1946).

It is assumed that the fluid under consideration is incompressible and homogeneous and in purely horizontal motion. With these assumptions solutions are obtained of the complete nonlinearized vorticity equation, both for a rotating plane and a rotating sphere. These solutions are very similar to those for the linearized form of the vorticity equations. However, in the plane case the assumed westerly current is now a function of the latitude, while for the linearized equations it could be assumed constant. Analogously, in the spherical case the angular velocity of the westerlies now depends on the latitude, while for the linearized equation it could be assumed as constant. *B. Haurwitz.*

Dufour, Louis. Sur les variations bariques et thermiques de l'atmosphère. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 93–100 (1946).

For part I cf. the same Bull. (5) 29, 374–380 (1943); these Rev. 7, 104.

Bouwkamp, C. J. A contribution to the theory of acoustic radiation. Philips Research Rep. 1, 251–277 (1946).

The propagation of sound in a homogeneous isotropic medium devoid of friction can be described by a velocity potential Φ when the amplitudes and the velocities of the particles in the medium are "small." This function Φ satisfies the wave equation (*) $\nabla^2\Phi - c^{-2}\partial^2\Phi/\partial t^2 = 0$, where c is the velocity of sound in the medium. The sound pressure p of the particles may be found from Φ by differentiation with respect to t, while the velocity $\mathbf{u}$ can be found by taking the gradient of Φ. The author studies the solutions of (*) for an infinite rigid plane which contains an aperture. On the rigid portion of the boundary, $\partial\Phi/\partial n = 0$, while in the aperture, $\partial\Phi/\partial n$ is considered known. In addition to continuity requirements on Φ, the Sommerfeld "Austrahlungsbedingung" must be satisfied at infinity [see Frank and von Mises, Die Differential- und Integralgleichungen der Mechanik und Physik, vol. 2, Vieweg, Braunschweig, 1935, Abschn. V]. The author assumes harmonic time variation of Φ and consequently eliminates the explicit appearance of the time variable.

The author then discusses the Rayleigh formula [Theory of Sound, vol. 2, London, 1896, sec. 278] and energy considerations which enable him to define some physically interesting parameters. These results are applied to a circular membrane oscillatory with azimuthal and radial nodal lines. The King theory of the circular disc is also discussed [L. V. King, Canadian J. Research 11, 135–155, 484–488 (1934)]. Some expansion theorems in this work are also considered. *A. E. Heins* (Pittsburgh, Pa.).

Possio, Camillo. L'influenza della viscosità e della conducibilità termica sulla propagazione del suono. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 274–292 (1943). [MF 16383]

In a perfect fluid a sonorous perturbation is propagated as a surface of discontinuity. The author finds that when viscosity (and also thermal conductivity) is taken into account, the disturbance is propagated as a layer which advances with the speed of sound but whose thickness increases as the square root of the time. Thus in air the thickness of the layer is about 2 cm. after 1 sec., during which time the disturbance has advanced 340 m. Within the layer the parameters which define the physical state of the fluid vary continuously. *L. M. Milne-Thomson.*

Tsien, Hsue-shen, and Schamberg, Richard. Propagation of plane sound waves in rarefied gases. J. Acoust. Soc. Amer. 18, 335–341 (1946).

Blokhintzev, D. The propagation of sound in an inhomogeneous and moving medium. I. J. Acoust. Soc. Amer. 18, 322–328 (1946).

Blokhintzev, D. The propagation of sound in an inhomogeneous and moving medium. II. J. Acoust. Soc. Amer. 18, 329–334 (1946).

Shirokov, M. F. The sound field of moving sound sources. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 494–496 (1945).

Hönl, H. Über das Schallfeld einer gleichförmig-translatorisch bewegten punktförmigen Schallquelle. Ann. Physik (5) 43, 437–464 (1943).

Consider a point sound source moving with constant velocity U in a medium for which the sound velocity is C. Let ϕ be the pressure variation. Replace $\partial/\partial t$ by $\partial/\partial t + U\partial/\partial x$ to bring the sound source to rest so that the governing equation is (1) $L(\phi) = \Delta\phi - c^{-2}(\partial/\partial t + U\partial/\partial x)^2\phi = 0$. Let (2) $\phi = u(x, y, z)\exp(i\omega t)$. The reviewer reformulates the contribution as follows. For subsonic U, the problem is essentially that of determining the Green's function where the "polar" singularity is fixed at the origin. This is symbolically the solution of (3) $L\phi = \delta(x, y, z)\exp(-i\omega t)$, where δ is the Dirac function. The solution is (4); u is the Fourier transform of $L(i\alpha_1, \cdots, i\alpha_n)^{-1}$ obtained by replacing $\partial/\partial x, \cdots$ by $i\alpha_1, \cdots$ and goes back at least to Zeilon. The author does not mention (3), but obtains the Green's function (4) by considering (5) $L(\phi) = f_\lambda(x, y, z)\exp(-i\omega t)$, where the f_λ's are decent functions subject to limiting values 0 away from the origin as $\lambda \to \infty$ and normalized so that the surface integrals containing the origin are all 1. Notwithstanding contrary claims by the author, the use of (5) is familiar to applied mathematicians and is recognized as the natural formulation of (3). The solution (4) is formal and becomes determinate on consideration of the integrals in the complex plane and proper choice of contour. [Actually, this was the primary concern of Zeilon's work, though the author refers only to Sommerfeld in this respect.] At least in the case of subsonic flow, this contour choice is determined by boundary conditions "at infinity" not explicitly written down in the author's initial formulation of the problem. For the supersonic case, the author verifies, as expected from the Huyghens construction, that the geometric confines of the wave front are given by the Mach cone. Furthermore, he shows that the solution is not bounded as one approaches the surface of this cone (and interprets this as indicating the limitations of the simple theory and the necessity of introducing shock considerations). The author considers also the cylindrical wave case. There are some interesting comments on differences between the two and three dimensional situations, particularly as regards phase consideration. [It would be interesting to know the connection with the familiar differences of residual effect between two and three dimensional wave problems, for presumably the latter situation has a physical formulation in terms of interference, that is to say, phase relationships. The author apparently overlooks Hadamard's work. For instance, the solution for the supersonic case is obviously connected with Hadamard's elementary solution. Moreover, the nonboundedness of the solution on approach to the Mach cone (characteristic conoid) is precisely the observation behind Hadamard's introduction of the "finite part of an integral."] *D. G. Bourgin* (Urbana, Ill.).

Wergeland, Harald. Some theoretical remarks on the diffraction of sound. Scattering by a rigid sphere. Avh. Norske Vid. Akad. Oslo. I. 1945, no. 9, 27 pp. (1945). [MF 16442]

The author derives the well-known result for the scattering of sound by a rigid sphere,

$$\psi_s = \sum_l \tfrac{1}{2}(2l+1)i^l P_l h_l(e^{2i\eta_l} - 1),$$

where $\tan\eta_l = -j_l'/n_l'$. He then considers the behavior of this result at high frequency. He shows that the term involving η_l gives rise to the uniform scattering exhibited by

a rigid sphere. The term independent of η_l gives rise to the shadow. The ratio of the amplitude shadow term to the incident wave is

$$-\rho \sin \tfrac{1}{2}\theta J_1(2\rho \sin \tfrac{1}{2}\theta) \int_{2kr \sin^2 \frac{1}{2}\theta}^{\infty} \tau^{-1} e^{i\tau} d\tau,$$

where $\rho = ka = 2\pi a/\lambda$, θ is the angle of scattering and r is the distance from the center of the sphere. *H. Feshbach.*

Elasticity, Plasticity

Kuzmin, R. O. On Maxwell's and Morera's formulae in the theory of elasticity. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 326–328 (1945).

Proof of the fact that both Maxwell's system of stress functions and Morera's system of stress functions represent the general solution of the differential equations of equilibrium, provided that the stress components and their derivatives are integrable. *E. Reissner.*

Pailloux, H. Transformation des équations de l'équilibre élastique et des vibrations. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 117–121 (1946).

The equations of elastic equilibrium (both static and dynamic) of an isotropic solid are transformed into equations for each component of the displacement rather than for the usual potentials. By the introduction of auxiliary functions whose Laplacians are the conventional functions, the problem is reduced to solving either Laplace or wave equations with auxiliary conditions. *G. F. Carrier.*

Marguerre, K. Bestimmung der Verzerrungsgrössen eines räumlich gekrümmten Stabes mit Hilfe des Prinzips von Castigliano. Z. Angew. Math. Mech. 21, 218–227 (1941). [MF 15853]

The paper begins with a discussion of the analogies and differences between and the meaning of the variational principle for the displacements and the variational principle for the stresses in the theory of elasticity. It is pointed out that the basic requirement for the existence of the latter principle is that the expressions for the strains must be taken in the form $\epsilon_x = \partial u/\partial x$, etc., and must not contain any nonlinear terms, so that the minimum principle for the stresses is less general than that for the displacements.

The variational principle for the stresses is then used to obtain a simple derivation of the appropriate stress-strain relations for space-curved beams. The difficulties which are avoided in this way are the geometrical considerations leading to the correct expressions for the curvature and torsion terms which must be related to the internal moments of the beam. *E. Reissner* (Cambridge, Mass.).

Costa de Beauregard, Olivier. Retour sur la dynamique et la thermodynamique des milieux continus. C. R. Acad. Sci. Paris 222, 1472–1474 (1946).

Vâlcovici, Victor. Sur une interprétation cinématique du tourbillon et sur la rotation des directions principales de la déformation. Mathematica, Timişoara 22, 57–65 (1946).

Carrillo, Nabor. Perturbation of a rigid circular field in an elastic field of uniform strength. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario 1944, 25–38 (1945). (Spanish)

Föppl, Ludwig. Die unendliche Halbebene bei beliebiger Randbelastung. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1941, 111–129 (1941).

A solution, in complex-variable form, of the problem of plane stress in the half plane. Explicit expressions are obtained for the stresses in the interior for the cases of a rectangular load curve and of a semicircular load curve. *E. Reissner* (Cambridge, Mass.).

Sokolovsky, W. W. Plastic equilibrium of a plane stressed state. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 209–220 (1946). (Russian. English summary) [MF 16845]

Sokolovsky, W. Plastic plane stressed state according to Saint-Venant. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 421–424 (1946).

The two papers are concerned with statically determinate states of plane stress in a medium for which the yield condition of Saint Venant holds. This yield condition assumes different forms according to whether the nonvanishing principal stresses are of equal or opposite signs:

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = \text{constant} = \sigma_0^2, \qquad \sigma_x\sigma_y \leqq \tau_{xy}^2,$$
$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = (2\sigma_0 - |\sigma_x + \sigma_y|)^2, \qquad \sigma_x\sigma_y \geqq \tau_{xy}^2.$$

The first form of the yield condition is fulfilled by setting

$$\sigma_x = A + \tfrac{1}{2}\sigma_0(2\omega + \cos 2\varphi), \quad \sigma_y = A + \tfrac{1}{2}\sigma_0(2\omega - \cos 2\varphi),$$
$$\tau_{xy} = \tfrac{1}{2}\sigma_0 \sin 2\varphi,$$

the second by setting

$$\sigma_x = \sigma_0\{k(1-\omega) + \omega \cos 2\varphi\}, \quad \sigma_y = \sigma_0\{k(1-\omega) - \omega \cos 2\varphi\},$$
$$\tau_{xy} = \sigma_0\omega \sin 2\varphi,$$

where $k=1$ if both σ_x and σ_y are positive and $k=-1$ if they are both negative. In each case the equations of equilibrium lead to two partial differential equations of the first order in ω and φ. In the first case this system is of hyperbolic type, in the second case, of parabolic type. As an example the author investigates the stresses in an infinite plastic slab with a circular or elliptic hole, the stresses at infinity being $\sigma_x = \sigma_y = \sigma_0$, $\tau_{xy} = 0$, the contour of the hole being subjected to a given normal stress. *W. Prager* (Providence, R. I.).

Sokolovsky, W. Plastic plane stressed states according to Mises. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 175–178 (1946).

The paper is concerned with statically determinate states of plane stress in a plastic medium for which the yield condition of von Mises is valid,

$$\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \text{constant} = \sigma_0^2.$$

This yield condition is fulfilled by setting

$$\sigma_x = 3^{-\frac{1}{2}}\sigma_0(3^{\frac{1}{2}} \cos \omega - \sin \omega \cos 2\varphi),$$
$$\sigma_y = 3^{-\frac{1}{2}}\sigma_0(3^{\frac{1}{2}} \cos \omega - \sin \omega \cos 2\varphi),$$

$\tau_{xy} = 3^{-\frac{1}{2}}\sigma_0 \sin \omega \sin 2\varphi$, where φ denotes the angle between the direction of the greatest principal stress and the x-axis. If these expressions for the stress components are introduced into the equations of equilibrium, simultaneous quasilinear partial differential equations of the first order are obtained for ω and φ. Contrary to the case for plane plastic strain, this system need not be of hyperbolic type. The integration of the system is discussed in the special case where it is of hyperbolic type ($\pi/6 < \omega < 5\pi/6$). *W. Prager.*

Sneddon, Ian N. A note on the equations of plane plastic flow. Philos. Mag. (7) 36, 629–635 (1945). [MF 16979]

The well-known complex presentation of two-dimensional fields of stress is applied to two-dimensional problems in plasticity. It is shown how typical fields of plastic stress can be obtained from the general formulas of the theory.

W. Prager (Providence, R. I.).

Torre, C. Über den plastischen Körper von Prandtl. Zur Theorie der Mohrschen Grenzkurve. Österreich. Ing.-Arch. 1, 36–50 (1946).

The author discusses the relation between the geometric representations of the yield condition by a curve in the (s_1, s_3)-plane (s_1 the maximum principal stress, s_3 the minimum principal stress) and by the envelope of the Mohr circles corresponding to critical states of stress. The yield condition $(s_1-s_3)^2+a(s_1+s_3)=b^2$, where a and b are constants, is shown to be sufficiently flexible for most practical purposes. This yield condition is used in the discussion of the stresses in a thick-walled plastic tube under internal and external pressures and the stresses in a compressed plastic wedge.

W. Prager (Providence, R. I.).

Drach, Jules. Sur la théorie des corps plastiques et l'équation d'Airy-Tresca. C. R. Acad. Sci. Paris 223, 461–464 (1946).

The determination of stresses and velocities of a perfectly plastic body in a state of plane stress is discussed. The author does not seem to be aware of the much more complete results of H. Geiringer [Fondements mathématiques de la théorie des corps plastiques isotropes, Mémor. Sci. Math., no. 86, Paris, 1937].

W. Prager.

Colonnetti, Gustave. Théorie de l'équilibre des corps élasto-plastiques. Bull. Tech. Suisse Romande 67, 82 pp. (1941).

Lectures at the University of Lausanne.

Charrueau, André. Sur la théorie des milieux continus en équilibre limite et sur la théorie des voiles minces. J. Math. Pures Appl. (9) 23, 77–89 (1944). [MF 15742]

The author considers the system

$$\partial z_1/\partial x+\partial z_2/\partial y=X, \tag{1}$$

$$\partial z_2/\partial x+\partial z_3/\partial y=Y, \tag{2}$$

$$F(x, y, z_1, z_2, z_3)=0, \tag{3}$$

where $X(x, y)$, $Y(x, y)$, $F(x, y, z_1, z_2, z_3)$ are known functions of their respective variables and z_1, z_2, z_3 are unknown functions of x, y. The first two relations represent the equilibrium equations. Equation (3) can be interpreted in two ways: as the yield condition in a plane plasticity problem or as the equation of the surface of a shell in the theory of elasticity. By eliminating z_3 and its derivative, the author obtains a system of two first order partial differential equations. If z_1, z_2 are known along some curve C (which is not a characteristic of this system) then one may obtain a power series solution for z_1, z_2, valid in some neighborhood for every point on C. If C is a characteristic, two cases arise. One case is shown to lead to no solution; the other case leads to an indeterminate solution. Finally, the author applies the theory to a generalized Coulomb yield condition and proves that the characteristics coincide with the slip lines. Furthermore, for the elasticity case, it is shown that the characteristics are the projections on the (x, y)-plane of the asymptotic curves of the shell surface.

N. Coburn.

Lehnickiĭ, S. G. On complex parameters occurring in general formulas of certain problems of the theory of elasticity of anisotropic solids. Leningrad State Univ. Annals [Uchenye Zapiski] 87 [Math. Ser. 13. Mechanics], 167–171 (1944). (Russian) [MF 16484]

The author has shown earlier [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 1, 78–90 (1937); 2, 181–210 (1938)] that the solution of plane problems in anisotropic elasticity and of the deflection of anisotropic elastic plates depends on the roots μ_i of the characteristic equation associated with the differential equation for Airy's function or for the deflection of the plate. The parameters μ_i depend on the elastic constants in the generalized Hooke's law and thus characterize the elastic properties of the medium. The paper contains a derivation of the laws of transformation of the parameters μ_i when the coordinate system is rotated through an angle φ about an axis normal to the plane of elastic symmetry of the medium.

I. S. Sokolnikoff.

Lehnickiĭ, S. G. Distribution of strain in rotation of an elliptic anisotropic plate. Leningrad State Univ. Annals [Uchenye Zapiski] 87 [Math. Ser. 13. Mechanics], 161–166 (1944). (Russian) [MF 16483]

A homogeneous anisotropic elliptical plate of constant thickness is revolved with constant angular velocity about an axis through the center, normal to the plane of the plate. The author obtains a simple solution for the mean stresses developed in the plate by rotation. The material of the plate is assumed to have one plane of elastic symmetry, parallel to the middle plane of the plate, and the displacements are taken sufficiently small to make the linear theory applicable.

I. S. Sokolnikoff (Los Angeles, Calif.).

Stevenson, A. C. Complex potentials in two-dimensional elasticity. Proc. Roy. Soc. London. Ser. A. 184, 129–179, 218–229 (1945). [MF 13353]

This paper deals with the problems of plane strain and generalized plane stress associated with isotropic homogeneous elastic media. Complex variables are employed. The author uses the combinations of stress components $\widehat{xx}+\widehat{yy}$ and $\widehat{xx}-\widehat{yy}+2i\,\widehat{xy}$, and expresses these combinations in terms of two complex potential functions $\Omega(z)$ and $\omega(z)$ (where $z=x+iy$) in such a manner that the equations of equilibrium and compatibility are satisfied automatically. The complex displacement $D=u+iv$ is also expressed in terms of these functions. Curvilinear coordinates ξ, η are introduced and the transformation from the above Cartesian stress combinations to the combinations $\widehat{\xi\xi}+\widehat{\eta\eta}$, $\widehat{\xi\xi}-\widehat{\eta\eta}+2i\,\widehat{\xi\eta}$ is found to be simple. This permits an easy consideration of boundary conditions.

The condition that the displacements are single valued or at most of the possible dislocational types is found to relate to the cyclic functions of the complex potential functions. Formulae are found for the force at the origin and the couple equivalent to the forces across a closed curve. These also involve the cyclic functions of the complex potentials and permit an easy consideration of nuclei of strain.

Various forms are assumed for the complex potential functions $\Omega(z)$ and $\omega(z)$. It is found upon examination that these yield the solutions to a great many problems, some of them apparently new. Typical are problems involving the concentric and eccentric ring spaces, an unstressed circular or elliptic hole in an infinite plate, nuclei of strain in an infinite plate and in a circular disc. Some of these problems involve body force and rotation with constant angular velocity, such as the case of a heavy circular disc in a verti-

cal plane with a general point fixed and the circular disc rotated in its own plane with constant angular velocity about a general point. *G. E. Hay* (Ann Arbor, Mich.).

Stevenson, A. C. Some boundary problems of two-dimensional elasticity. Philos. Mag. (7) **34**, 766–793 (1943).

The theory developed in the paper reviewed above (which was written in 1940) is developed in a somewhat more general form. Solutions to additional problems are presented, among which are the cases of an annulus under gravity and supported in various ways, an elliptic cylinder rotating about its axis with constant angular velocity, a semi-infinite plate with a trochoidal boundary and under tension at infinity and an infinite plate with a curvilinear polygonal hole. *G. E. Hay* (Ann Arbor, Mich.).

Stevenson, A. C. Tension of semi-infinite plate with notched boundary. Philos. Mag. (7) **36**, 178–183 (1945). [MF 13340]

The author applies the theory developed in the two papers reviewed above to a semi-infinite plate with its left-hand boundary given by the polar equation $r=c\sec\theta-\lambda c\cos\theta$, where $\lambda\leqq 1$. This curve has a cusp at the origin and a vertical asymptote to the left of the origin. The plate is subjected to a uniform tension at infinity and parallel to the asymptote. The two unknown functions $\Omega(z)$ and $\omega(z)$ are determined in closed form by mapping the region to the right of the boundary curve on the unit circle. *G. E. Hay.*

Mindlin, Raymond D. The analogy between multiply-connected slices and slabs. Quart. Appl. Math. **4**, 279–290 (1946).

This paper deals with the most general case of the mathematical analogy between the two dimensional field of stress (in a slice) and the transverse flexure of a thin plate (slab). The author derives the general boundary conditions for the slab when the slice is multiply-connected and is stressed by any combination of boundary loadings, body forces, dislocations and thermal dilatations. It is stated that the analogy has its most useful applications in the last three cases for which slice experiments are difficult while the analogous conditions for the slab, as developed in this paper, are easy to handle. *E. Reissner* (Cambridge, Mass.).

Dworzak, W. Der freie Rand an rechteckigen Platten. Österreich. Ing.-Arch. **1**, 66–77 (1946).

The author derives expressions for the deflection and stresses due to concentrated moments and transverse forces applied to an otherwise free edge of a thin elastic plate. In particular, he considers the rectangular plate with two opposite edges simply supported while the remaining two edges are free except for the concentrated loadings. The singular part of the solution is reduced to an expression in closed form. *E. Reissner* (Cambridge, Mass.).

Persen, Leif N. Exact solutions for bending of elliptic plates. Norske Vid. Selsk. Forh., Trondhjem **18**, no. 1, 1–4 (1945). (Norwegian)

The paper discusses thin elliptic plates with edges clamped along the entire contour. A deflection function

$$w=\sum_{i=2}^{\infty}A_i\{x^2/a^2+y^2/b^2-1\}^i$$

is assumed and the corresponding load function $p(x, y)$ computed. Some simple special cases are of interest and are worked out in detail. *P. Neményi* (Washington, D. C.).

Djanelidze, G. J. Determination of shearing forces in the bending of supported thin plates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 221–228 (1946). (Russian. English summary) [MF 16846]

The paper is concerned with the determination of the shearing forces produced by bending of thin plates, without previous calculation of deflections. It is shown that in the case of a freely supported plate, bounded by rectilinear contours, the problem is reduced to the solution of Poisson's equation $\Delta\psi=p(x, y)$ with $\psi=0$ on the boundary. The author calculates the shearing forces in square and right-angled isosceles triangular plates subjected to concentrated loads. The Schwarz-Christoffel transformation is used and the shearing forces are given in terms of elliptic functions. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Vlasov, V. Z. Building mechanics of rectangular thin plates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 173–192 (1946). (Russian. English summary)

A procedure for solving two-dimensional problems of the theory of elasticity for thin rectangular plates and thin prismatic shells is described. It is based on a reduction of the partial differential equation for deflection to a system of ordinary differential equations. The paper contains several worked examples illustrating the method. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Melan, E. Ein rotationssymmetrischer Spannungs- und Verzerrungszustand einer gelochten Scheibe bei nichtlinearem Spannungs-Dehnungsgesetz. Österreich. Ing.-Arch. **1**, 14–21 (1946).

The paper is concerned with the determination of stresses and strains in a circular disk with a central circular hole. Radial stresses are uniformly distributed over the cylindrical surface of the hole and the disk material is assumed to obey a nonlinear stress-strain relation which is obtained from Hooke's law for an incompressible elastic body by replacing Young's modulus by a suitable invariant of the strain tensor. Explicit expressions for strains and displacements are given for two choices of this invariant. *W. Prager.*

Friedman, M. M. Bending of a thin isotropic plate having an aperture. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 334–338 (1945). (Russian. English summary) [MF 15343]

A thin infinite isotropic plate contains an elliptic, triangular or rectangular hole with rounded corners. It is bent by forces and bending moments distributed along the boundary of the hole. The author uses the complex variable methods introduced by Muschelišvili to obtain the deflections and stresses. He also solves the problem for the case when the deflections and their normal derivatives are prescribed on the boundary of the hole. *I. S. Sokolnikoff.*

Holgate, S. The transverse flexure of perforated isotropic plates. Proc. Roy. Soc. London. Ser. A. **185**, 35–49 (1946). [MF 15184]

The author adopts the method used by A. E. Green and G. I. Taylor [same Proc. Ser. A. **184**, 181–195, 231–252 (1945); these Rev. **7**, 267] to determine the behavior of a large thin isotropic plate bent or twisted by couples at infinity. The plates are assumed to have holes defined by the transformation $dz/d\zeta=a_0e^{-i\zeta}+b_ne^{in\zeta}$, where $z=x+iy$ is in the plane of the plate, $\zeta=\xi+i\eta$ is real on the boundary of the hole, and a_0 and b_n are constants. A particular choice of a_0

and b_n yields holes of various shapes, and the author considers circular, elliptical, triangular and rectangular holes with rounded corners. Numerical calculations, leading to the determination of the stress concentration on the boundary of triangular and square holes, are given when the plate is bent and twisted by couples applied at infinity. [Apparently the author is not familiar with the papers of M. M. Friedmann [J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, no. 1, 93–102 (1941); these Rev. **3**, 30; cf. also the preceding review] which contain an elegant treatment of the problem. It appears to the reviewer that, instead of using the transformation given above, it is simpler to introduce a mapping function $z=a_0\zeta+b_n\zeta^{-n}$, which transforms the region bounded by the plate onto the exterior of the unit circle in the ζ-plane.]

I. S. Sokolnikoff (Los Angeles, Calif.).

Holgate, S. The transverse flexure of perforated aeolotropic plates. Proc. Roy. Soc. London. Ser. A. **185**, 50–69 (1946). [MF 15185]

The results of the paper reviewed above are extended to cover the case of thin anisotropic plates containing an elliptical hole and bent and twisted by couples applied at infinity. Numerical calculations are carried out for the distribution of stress couples along the edge of an elliptical hole in a spruce plate (for one case of cylindrical bending) and for a spruce plate containing a circular hole and subjected to the action of bending and twisting couples at infinity. [The reviewer calls attention to the work of S. G. Lechnitzky, J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **2**, 181–210 (1938); **5**, 71–91 (1941); these Rev. **3**, 30], who gave a comprehensive theoretical treatment of the problem of deflection of thin anisotropic plates.]

I. S. Sokolnikoff (Los Angeles, Calif.).

Tranter, C. J. On the elastic distortion of a cylindrical hole by a localised hydrostatic pressure. Quart. Appl. Math. **4**, 298–302 (1946).

The stresses and displacements caused by the application of a hydrostatic pressure over a small part of the length of a cylindrical hole in an infinite elastic solid differ considerably from those that result from the application of the same pressure over the entire length of the hole. H. M. Westergaard [von Kármán Anniversary Volume, pp. 154–161, California Institute of Technology, Pasadena, 1941; these Rev. **3**, 32] discussed the problem by an approximate method. In the present note an exact solution is given. The method is a modification of that given by A. W. Rankin [J. Appl. Mech. **11**, A77–A85 (1944)] for the similar problem of a band of uniform pressure applied to a long cylindrical shaft. The calculation of the maximum radial displacement in two special cases shows that for these cases the approximate method yields results that are too large by about twenty per cent. *H. W. March* (Madison, Wis.).

Lourye, A. I. Concentration of stresses in the vicinity of an aperture in the surface of a circular cylinder. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 397–406 (1946). (Russian. English summary)

Sneddon, I. N. The distribution of stress in the neighbourhood of a crack in an elastic solid. Proc. Roy. Soc. London. Ser. A. **187**, 229–260 (1946).

Sneddon, I. N., and Elliott, H. A. The opening of a Griffith crack under internal pressure. Quart. Appl. Math. **4**, 262–267 (1946).

Sneddon, I. N. The elastic stresses produced in a thick plate by the application of pressure to its free surfaces. Proc. Cambridge Philos. Soc. **42**, 260–271 (1946).

Using the method of Hankel transforms, Harding and the author have shown [same Proc. **41**, 16–26 (1945); these Rev. **6**, 251] that the stresses and displacements of a symmetrically loaded thick plate of infinite extent can be determined with the aid of a function $G(\xi, Z)$ satisfying (1) $(d^2/dz^2-\xi^2)^2G=0$, where Z is the axis of symmetry normal to the plate. The unknown constants in the solution of (1) are determined by the boundary conditions. For a plate of infinite thickness, the problem reduces to that of Boussinesq; the author shows that his results are identical with those of K. Terazawa [J. Coll. Sci. Tokyo Imp. Univ. **37**, no. 7 (1916)]. For loadings that are symmetrical with respect to the thickness of the plate, there is considerable simplification in the calculation. However, even then the factor $f=u/(u+\sinh u)$ in the integrals makes the computation difficult. The author replaces this factor by $f_1=(\alpha u+\frac{1}{2})e^{-\nu u}+2ue^{-u}$ with constants α and ν so that the area under the curve f_1-f is as small as possible. This procedure is found to be satisfactory for loads concentrated in a small area of the surface of the plate. Numerical calculations show that the error for the case of concentrated loads is only 1%. Calculations for loads uniformly distributed within a small circle are also carried out. *H. S. Tsien.*

Green, A. E., and Hearmon, R. F. S. The buckling of flat rectangular plywood plates. Philos. Mag. (7) **36**, 659–688 (1945). [MF 16974]

Plywood of usual construction may be treated as an orthotropic material. If the equation of buckling of a flat plate of plywood is written for rectangular axes inclined to the orthotropic axes it will be observed that it has the same form as that written for a plate of any aelotropic material. Methods are developed for finding the buckling loads of rectangular plates of plywood having the grain of the face plies at given angles to the edges and subject to various edge conditions. The loading is either uniform compression along two opposite edges or uniform shear along all edges. Buckling loads are obtained either with the aid of the differential equation of buckling or by an energy method. For plates with simply supported edges the buckled surface is represented by a double Fourier sine series with coefficients A_{rs}. The difficulties arising from the presence of the derivatives $\partial^4w/\partial x^3\partial y$ and $\partial^4w/\partial x\partial y^3$ in the differential equation of buckling are met by expanding terms of the form $\cos mx \cos my$ in double Fourier sine series. An infinite system of homogeneous equations linear in the constants A_{rs} is obtained. As in previous similar analyses these equations are found to separate into two groups. For a nonvanishing solution of each group the corresponding infinite determinant must vanish. The smaller of the two loads found by equating these determinants to zero is the buckling load. In certain instances selection of a small number of elements of the determinants leads to an approximate general formula. In others the approximate evaluation of the buckling load for each case from a limited number of elements of the determinants is necessary. For plates with clamped edges the application of an energy method leads to a similar infinite system of equations. Some other boundary conditions are briefly considered. Numerical results are given. The authors state that many of the results of the paper are new but that other writers have treated problems of the buckling of plywood plates in confidential reports issued

during the war. In a footnote added in proof references are given to such reports which were available to the authors and from which restrictions have been removed. Reference might also have been made to two papers by E. Seydel on the buckling of isotropic and orthotropic plates in shear [Ing.-Arch. **4**, 169–191 (1933); Z. Flugtechnik Motorluftschiffahrt **24**, 78–83 (1933); translation of the latter, Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **705** (1933)].
H. W. March (Madison, Wis.).

Budiansky, Bernard, and Hu, Pai C. The Lagrangian multiplier method of finding upper and lower limits to critical stresses of clamped plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1103**, 33 pp. (1946).

Korenev, B. G. Method of initial parameters in problems of circular plates and shells of revolution. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 165–172 (1946). (Russian. English summary) [MF 16843]

The author applies the method of initial parameters developed by A. N. Krylov [Vibration of Ships, Leningrad, 1936] to the following problems: (a) flexural vibrations of a circular plate loaded by forces of the form $P(r)\cos n\varphi \cos kt$, where r and φ are the polar coordinates in the plane of the plate, n and k are positive integers and t is time; (b) deflection of a plate resting on an elastic foundation and loaded by forces of the form $g(r)\cos n\varphi$; (c) a thin circular conical shell of constant thickness loaded by forces symmetrically distributed about the axis of the cone. These problems have already been considered by other investigators, but the solutions given by the author require less involved calculations.
I. S. Sokolnikoff (Los Angeles, Calif.).

Novožilov, V. V. New method for the calculation of thin shells. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 35–48 (1946). (Russian)

The author adjoins to the classical Kirchhoff-Love equations of thin shells three equations of compatibility deduced by A. Goldenweiser [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **4**, 35–42 (1940); **8**, 3–14 (1944); these Rev. **6**, 251] and derives a system of six differential equations (of eighth order) for the stresses. If Poisson's ratio μ is set equal to zero, the system becomes symmetric with respect to a combination of stresses, so that by introducing new dependent variables it can be reduced to a system of three equations of the fourth order. The resulting equations call for no simplifications beyond those implicit in the Kirchhoff-Love theory. By neglecting, in the equations of equilibrium and compatibility, certain terms of the order δ/R (δ, thickness of the shell; R, radius of curvature) the original eighth-order system can be reduced to a fourth-order system when $\mu\neq 0$. This latter result can be viewed as a generalization of Meissner's theory of symmetric deformation of shells of revolution.

The author states that his formulation of the problem greatly simplifies the solution of several classes of problems in shell theory and promises to publish soon his results on pipes of arbitrary cross-section and on shells of revolution of arbitrary form. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Novožilov, V. V. The calculation of cylindrical shells. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Acad. Nauk SSSR] **1946**, 803–816 (1946). (Russian)

The paper is given to the illustration of the method of solution of problems on thin elastic shells, proposed by the author in the paper reviewed above. The following problems are considered: (a) circular cylindrical plates and circular cylinders with freely supported ends, (b) deformations of pipes whose cross-sections have two mutually orthogonal axes of symmetry. *I. S. Sokolnikoff.*

Novožilov, V. V. Calculation of shells—bodies of revolution. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 949–962 (1946). (Russian)

The author applies his formulation of the general theory of thin shells [cf. the second preceding review] to shells of revolution. He deduces as special cases the well-known results of E. Meissner on symmetrically loaded shells of revolution and E. Schwerin's results on the wind loading of domes. An interesting new result is that the problems of wind loading of shells of revolution can be reduced to the integration of a second-order linear differential equation. The paper contains an illustration of the application of the theory to spherical and catenoidal shells. [For cylindrical shells cf. the preceding review.] *I. S. Sokolnikoff.*

Rabotnov, J. N. Local stability of shells. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 111–112 (1946).

Biezeno, C. B., and Koch, J. J. The effective width of cylinders, periodically stiffened by circular rings. Nederl. Akad. Wetensch., Proc. **48**, 147–165 (1945). [MF 15919]

This paper is concerned with the analysis of the stresses in a thin cylindrical shell, periodically stiffened by rings, and acted upon by loads applied to the rings in their own plane. The question is what fraction of the shell is to be added to the rings, as flanges of a fictitious T-section, so that the fiber stress in the fictitious flanges (the width of which is called "effective width") equals the circumferential stress in the reinforced shell at the junction of shell and stiffening ring. The authors state that straightforward analysis of this problem is very involved and that the present analysis, which makes use of some approximations, is much simpler and leads to results of sufficient accuracy. Numerical results are presented in the form of extensive diagrams and tables for each harmonic of the loads, for various values of the ratio of distance between rings to shell radius, and for various values of the ratio of shell thickness to shell radius. *E. Reissner* (Cambridge, Mass.).

Mandel, J. La torsion plastique. Ann. Ponts Chaussées **1946** (116ᵉ année), 1–33 (1946).

A general mathematical formulation of the problem of plastic torsion of cylindrical bars with arbitrary cross section is attempted in the case where the shearing stress depends on the shear strain and the rate of shear. The z-axis of a system of rectangular Cartesian coordinates x, y, z is taken parallel to the generators of the cylinder and the displacement components are assumed to be of the form $u=-\theta yz$, $v=\theta xz$, $w=\theta\varphi(x, y)$, where $\theta=\theta(t)$ denotes the angle of twist per unit length. [The author fails to prove that this kinematical scheme has the necessary generality; actually, the warping function φ will be independent of the time t only if the shearing stress depends linearly on shear strain and rate of shear [see, for instance, the reviewer's Theory of Plasticity, Brown University, Providence, 1942, chap. 9].] Plastic torsion of thin-walled open sections and tubes is discussed at some length and the results are specialized to the case of an elastic-plastic material which yields under constant shearing stress. *W. Prager.*

Kuzmin, R. O. Concerning the torsion of homogeneous isotropic cylinders. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 11–12 (1946).

The torsional stiffness of a cylindrical prism of cross section S is given by

$$C=\mu\iint_S(x^2+y^2)dxdy-\mu\iint_S(x\partial\psi/\partial x+y\partial\psi/\partial y)dxdy,$$

where ψ is a harmonic function which attains the value $\frac{1}{2}(x^2+y^2)$+constant on the boundary of S. The author shows that $C=\mu(S_1-S_2)$, where S_1, S_2 are the areas when S is mapped conformally by $w_1=\frac{1}{2}z^2$ ($z=x+iy$) and also by $w_2=\omega(z)$, where $\omega(z)$ is such a function that $\Re\omega(z)=\frac{1}{2}|z|^2$ on the boundary of S. Applications of this result are made to cross sections of regular polygons of sides $n=3, 4, 5, \cdots$.

D. L. Holl (Ames, Iowa).

von Kármán, Theodore, and Chien, Wei-Zang. Torsion with variable twist. J. Aeronaut. Sci. 13, 503–510 (1946).

The authors study the torsion of thin walled cylindrical structures stiffened with an internal web system and subjected to a constraint causing nonuniform axial twist. The problem is to determine the rate of twist and the warping of the various sections when the bending stiffness of the thin wall is neglected. It is shown that the warping or axial displacement satisfies a linear homogeneous integro-differential equation which in the case of thin walled polygonal cross sections reduces to Laplace's equation and certain transition conditions at the corner points. In particular, the twist is uniform in a regular polygon and agrees with the Saint-Venant theory. The cases of a rectangular box column and a symmetrical dumbbell cross section are carried out in detail.

D. L. Holl (Ames, Iowa).

Galin, L. A. Pressure of a punch with friction and cohesion domains. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 413–424 (1945). (Russian. English summary) [MF 15349]

A rigid punch with plane base $ABCD$ is pressed against an elastic half plane $y<0$ by a force P. Friction of magnitude ρ acts along the segments AB and CD of the punch, but the tangential stress is insufficient to produce sliding along the segment BC, so that the punch adheres to the medium along BC. A solution of this elastic problem is given with the aid of two functions of a complex variable, analytic in the region $y<0$, which are determined from integrals of Cauchy's type. The following boundary conditions are used for the determination of these functions. The displacements are $u(x, 0)$=constant on BC and $v(x, 0)$=constant along AB, BC and CD. The tangential stress $t(x)$ and the normal stress $p(x)$ on the boundary satisfy the conditions $t(x)-\rho p(x)=0$ on AB, $t(x)+\rho p(x)=0$ on CD, $t=p=0$ elsewhere. It is shown that the ratio of the length of the range of adhesion to the total length of the punch depends on ρ and the ratio of Poisson. [Cf. the following review.]

I. S. Sokolnikoff (Los Angeles, Calif.).

Falkovich, S. V. Pressure of a rigid punch on an elastic semi-plane with ranges of sliding and adhesion on the line of contact. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 425–432 (1945). (Russian. English summary) [MF 15350]

A rigid punch with plane base $ABOCD$, of length $AD=2b$, is in contact with the elastic half-plane $y<0$ along the portion $-b<x<b$ of the x-axis. The punch is pressed by a force P, acting along the y-axis and passing through the middle of the punch at O. The punch adheres to the half-plane along the portion BC of length $2a$, but the contact along AB and CD is assumed to be sliding and frictionless. The elastic problem is reduced to the determination of two functions of a complex variable, regular in the region $y<0$. The solution is given with the aid of elliptic functions and the formulas for normal and tangential stresses along the line of contact are obtained. [Cf. the preceding review.]

I. S. Sokolnikoff (Los Angeles, Calif.).

Reissner, Eric. Analysis of shear lag in box beams by the principle of minimum potential energy. Quart. Appl. Math. 4, 268–278 (1946).

The effect of the deformability of the thin sheet coverings of box beams, which is neglected in the elementary beam theory, is investigated. The minimum potential energy method used gives simpler results than those obtained earlier by the author using the theorem of least work [J. Aeronaut. Sci. 8, 284–291 (1941); these Rev. 3, 96]. The basic step is an assumption as to the spanwise distribution of the sheet cover displacements which leads, via the calculus of variations, to a differential equation for the box beam deflection. This equation is analogous to the classical beam equation but is of sufficiently higher order to allow the satisfaction of boundary conditions pertaining to the sheet covering.

G. F. Carrier (Providence, R. I.).

Reissner, E., and Thomas, G. B. Note on the shear stresses in a bent cantilever beam of rectangular cross section. J. Math. Phys. Mass. Inst. Tech. 25, 241–243 (1946).

Lévi, Robert. Étude générale du flambement des arcs. C. R. Acad. Sci. Paris 220, 436–438 (1945). [MF 13957]

The author considers the buckling of plane arches with small initial curvature. The initial shape of the central line is expressed as a Fourier series and the deflection is also deduced as a Fourier series. [Cf. S. Timoshenko, Theory of Elastic Stability, McGraw-Hill, New York, 1936, pp. 31–32.]

G. E. Hay (Ann Arbor, Mich.).

Ratzersdorfer, J. A buckling problem. A beam resting on a continuous elastic foundation with concentrated elastic end supports. Aircraft Engrg. 18, 266–268 (1946).

Levina, C. O. Complementary study of stresses in the room separating pillars. Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog. no. 108, 43 pp. (1941). (Russian. English summary)

Ledoux. Application de la théorie des surfaces à l'étude des déformations des hélices aériennes. Ann. Univ. Lyon. Sect. A. (3) 4, 5–43 (1941).

Lubkin, S., and Stoker, J. J. Corrections to our paper "Stability of columns and strings under periodically varying forces." Quart. Appl. Math. 4, 309–310 (1946).

The paper appeared in the same Quart. 1, 215–236 (1943); these Rev. 5, 83.

Sneddon, I. N. The elastic response of a large plate to a Gaussian distribution of pressure varying with time. Proc. Cambridge Philos. Soc. 42, 338–341 (1946).

The author's summary is as follows. The analysis for a general symmetrical distribution of applied pressure was given in a recent paper [Sneddon, same Proc. 41, 27–43 (1945); these Rev. 6, 229], the case of a uniform pressure

over a circle being discussed in some detail. In the present note an account is given of the vibrations produced in a thin elastic plate by a Gaussian distribution of pressure, varying with time, and applied to one of the free surfaces. A relation is found between the applied pressure and the displacement and velocity of any point of the plate. The usual assumptions of the elastic theory of thin plates are made, the displacement of the plate as a whole being inferred from that of its central surface. It is assumed that the pressure is applied symmetrically so that cylindrical polar coordinates are used; at any instant the applied pressure is taken to be proportional to $\exp(-r^2/s^2)$, where r denotes distance from the origin of coordinates and s is some constant distance. As an example of the use of the results derived, the motion produced when the pressure is due to an impulse of very short duration is considered briefly.
H. S. Tsien (Cambridge, Mass.).

Conway, H. D. The calculation of frequencies of vibration of a truncated cone. Development of the theories for the longitudinal, flexural and torsional vibration of a truncated wedge. Aircraft Engrg. **18**, 235–236 (1946).

The eigen-frequencies and functions associated with a truncated cone rigidly supported at its large end and free at the other are investigated. A strictly one-dimensional analysis is applied for longitudinal, flexural, and torsional vibrations. The solutions are thus valid only for cones which are very slender and whose taper is very small (that is, $r_{av}/l \ll 1$; $\partial r/\partial l \ll 1$). The eigen-functions appear in the form of Bessel functions of orders $\frac{1}{2}$, 2, $\frac{3}{2}$, respectively.
G. F. Carrier (Providence, R. I.).

Issakovitch, M. A. Sur localisation de l'énergie potentielle dans une corde vibrante. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 95–98 (1946).

It is shown that, to the first order of approximation, the tension (and thus the potential energy) in a string undergoing essentially transverse vibrations is a function of the time only. From this are deduced the known form for the tension and for the induced longitudinal motion. [For more detailed results, cf. Carrier, Quart. Appl. Math. **3**, 157–165 (1945); these Rev. **7**, 13.] *G. F. Carrier.*

Caloi, Pietro. Nuovo metodo per determinare le coordinate ipocentrali e le velocità di propagazione delle onde longitudinali e trasversali dirette. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 355–358 (1943).

Scherman, D. I. On diffraction of elastic waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 626–629 (1945). [MF 16642]

The author considers the problem of steady vibrations when an elastic medium fills a finite simply connected region S, bounded by a suitably "smooth" curve L. The determination of the translation vectors in S from their values on L amounts to suitable resolution of equations $\Delta\phi+k_1^2\phi=0$, $\Delta\psi+k_2^2\psi=0$ (Laplacian Δ), where k_1, k_2 are constants. The problem is finally reduced to a Fredholm system of integral equations. *W. J. Trjitzinsky* (Urbana, Ill.).

Gogoladze, V. G. Elastic movements in a medium with elastic after working (hereditary). Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog. no. **109**, 24 pp. (1941). (Russian)

Gogoladze, V. G. Motion of seismic energy in different media. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 554–555 (1945).

Carrier, G. F. The propagation of waves in orthotropic media. Quart. Appl. Math. **4**, 160–165 (1946). [MF 16961]

The methods of a previous paper [same Quart. **2**, 31–36 (1944); these Rev. **6**, 26] are extended to discuss the displacement potentials of dynamic phenomena in orthotropic media. Consideration is limited to media which are isotropic in planes parallel to one of the planes of elastic symmetry and for which a certain relation among the elastic constants is satisfied. The number of independent elastic constants of such media is thereby reduced to four. *H. W. March.*

Gubanov, A. Rayleigh waves at the solid-liquid boundary. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **15**, 497–502 (1945). (Russian. English summary) [MF 15384]

The author investigates surface waves of the Rayleigh type on the boundary of a solid half-space under the assumption that the complementary half-space is filled by (i) moving perfect fluid, (ii) viscous fluid at rest. In the first case he concludes that the effect of the fluid on the Rayleigh waves becomes appreciable only if the velocity of the fluid becomes comparable with that of the Rayleigh waves. In the second case he shows that the damping effect of viscosity is too insignificant to be of interest for the theory of seismic waves at the bottom of the ocean. *L. Bers.*

MATHEMATICAL PHYSICS

Quantum Mechanics

Bass, Jean. Quelques remarques sur la signification du théorème des probabilités composées dans le formalisme de la mécanique quantique. C. R. Acad. Sci. Paris **222**, 1372–1374 (1946).

If a and b are chance variables associated with Hermitian linear transformations A and B of n-space, the correlation between a and b is defined only when A and B commute. If A and B have n and p simple characteristic values, respectively, the author defines two Hermitian linear transformations A' and B' of np-space, which commute even if A and B do not, such that A' has the characteristic values of A as p-fold characteristic values and B' has the characteristic values of B each as n-fold characteristic values. In terms of A' and B' a generalized notion of the correlation between a and b is definable, which is in general, however, not uniquely determined by A and B. This notion is then extended to Hilbert space and to Hermitian operators in Hilbert space with either discrete or continuous spectrum. An example is worked out in which A is the operator with continuous spectrum associated with the coordinate x and B has just two simple characteristic values, being associated with a spin coordinate. *O. Frink* (State College, Pa.).

Slansky, Serge. Sur certaines généralisations des changements de coordonnées. C. R. Acad. Sci. Paris **222**, 857–858 (1946). [MF 16284]

The author notes that it is meaningful in quantum mechanics to use formulas for a change of coordinates which involve linear operators at places where there would ordinarily be numerical coefficients. For example, one may ex-

pand a wave function, to which such a change of variables is to be applied, in a series of eigenfunctions of the operator and then replace the operator by its corresponding eigenvalues, which are numerical. This device allows him to define the partial derivative of a wave function with respect to a linear operator having a continuous spectrum. In special cases the definition is shown to be consistent with well-known formulas. *O. Frink* (State College, Pa.).

Slansky, Serge. La mécanique ondulatoire relativiste des systèmes et les transformations de M. Destouches. Cahiers de Physique no. 21, 1–11 (1944).

The author discusses some transformations proposed by J. L. Destouches [J. Phys. Radium (7) 8, 145–152 (1937)] and a modification of these transformations which are said to have an application to the relativistic treatment of the n-particle problem in quantum mechanics. In these transformations the coordinates of a particle are considered to be operators in some other space which is not clearly defined and does not seem closely associated with the problem.

The author proves that the $(3n+1)$-"dimensional" configuration space may be decomposed in an invariant manner into a four-"dimensional" space and a $(3n-3)$-"dimensional" space under the transformations considered. Moreover, the transformation induced in the four-"dimensional" space is a Lorentz transformation. The "points" of all these spaces are also operators in the unspecified space mentioned above. *A. H. Taub* (Seattle, Wash.).

Cazin, Michel. Extension du calcul vectoriel adaptée à la mécanique ondulatoire. C. R. Acad. Sci. Paris 222, 992–994 (1946). [MF 16391]

Cazin, Michel. La dérivation symbolique en calcul vectoriel gauche. C. R. Acad. Sci. Paris 222, 1079–1081 (1946). [MF 16525]

Cazin, Michel. La composition des mouvements en cinématique opératorielle. C. R. Acad. Sci. Paris 222, 1207–1209 (1946). [MF 16730]

These notes deal with an extension of vector analysis to vectors whose components are noncommutative quantities, such as the coordinate and momentum operators of quantum mechanics. The first note is concerned with vector algebra, the second with a generalized notion of the time-derivative of a vector and the third with quantum kinematics and, in particular, with the laws of composition of velocities and accelerations. Complications are caused by the fact that a vector has distinct left and right components in terms of a set of orthogonal unit vectors, because of the failure of commutativity. *O. Frink* (State College, Pa.).

Gião, Antonio. Le problème cosmologique généralisé et la mécanique ondulatoire relativiste. Portugaliae Phys. 2, 1–98 (1946).

A nonarbitrary mathematical scheme (n.-a.m.s.) is defined to be such that the "content" of the scheme, that is, the set of functions "contained" in it, determines completely the intrinsic properties of structure and form of the mathematical framework, or "container," and vice versa. Every n.-a.m.s. is postulated to be the basis of some physical existence. The author considers Riemannian N-dimensional spaces e_N immersed in an e_{N+1} and claims that there is only one n.-a.m.s. among these subspaces. The metric for e_{N+1} is (1) $d\Sigma^2=\Gamma_{\mu\nu}dX^\mu dX^\nu$ and the internal and external metric forms for e_N are, respectively, (2) $ds^2=g_{ik}dx^idx^k$ and (3) $d\Omega^2=\omega_{ik}dx^idx^k$, where $\omega_{ik}=X^\mu_{;ik}n_\mu$, n^μ being the unit vector normal to e_N in e_{N+1} and $X^\mu_{;ik}$ the second tensor derivative of X^μ with respect to the x's. Intrinsic properties of (2) and (3) are taken to be represented by symmetric tensors of the second order satisfying the usual conservation laws. For familiar reasons the equations

$$(4)\quad R_{ik}-\tfrac{1}{2}g_{ik}(R+\lambda_g)=\kappa_gT_{ik},\quad (5)\quad S_{ik}-\tfrac{1}{2}\omega_{ik}(S+\lambda_\omega)=\kappa_\omega U_{ik},$$

corresponding to (2) and (3), respectively, are considered. A simple comparison of the number of equations (including those of Gauss and Codazzi) with the number of unknowns is taken to imply that $N=4$ and that (2) is of class one. Eigenfunctions (column matrices) Ψ^n of the familiar operator equation (6) $\epsilon^i\partial\Psi/\partial\rho^i=-\sqrt{(\alpha)}\Psi$ are used to express T^{ik} in the form

$$(7)\quad \Psi_n{}^\dagger(\epsilon^i)_n\frac{\partial\Psi^n}{\partial\rho_k}-\frac{\partial\Psi_n{}^\dagger}{\partial\rho_k}(\epsilon^i)_n\Psi^n+\Psi_n{}^\dagger(\epsilon^k)_n\frac{\partial\Psi^n}{\partial\rho_i}-\frac{\partial\Psi_n{}^\dagger}{\partial\rho_i}(\epsilon^k)_n\Psi^n$$

and similarly for Φ_n and U^{ik}. Here ρ_i are local orthogonal geodetic coordinates and, for each n, ϵ^i are 4×4 matrices satisfying $\epsilon^i\epsilon^k+\epsilon^k\epsilon^i=2\delta^{ik}I$. Metrics (2) and (3) are both found to be hyperbolic normal and the author discusses, in the function- and number-content of the n.-a.m.s., the usual scalars, pseudo-scalars, polar and axial vectors, and the antisymmetric tensors arising from the ϵ^i as bilinear forms in the Ψ^n and in the Φ^n. An approximate solution of (4), (5), etc. gives a space-time of de Sitter-Lanczos type with constant mean curvature and it is concluded that an accurate solution would yield a natural unit of length.

The physical interpretation correlates T^{ik} with the energy-momentum tensor of matter, the ω_{ik} with the description of the electromagnetic field, and the above-mentioned bilinear forms with various mass- and charge-current density vectors, mechanical and electromagnetic moment tensors, etc. In this interpretation the relations (8) $\omega_{ik}=\chi g_{ik}$, where χ is a constant corresponding to constant mean curvature, are essential.

The elementary corpuscles of the universe have an enumerable infinity of increasing positive masses and of decreasing negative masses. Similarly for the elementary charges in the universe. Only the numerically smallest contribute appreciably to their respective metrics. The rays of electromagnetic radiation are null geodesics for (3), and also for (2) provided (8) holds. The constants c and h are introduced in a "nonarbitrary" manner, neutrons and protons occur as close unions of electrons, and mesons are incomplete protons and neutrons.

A cosmological wave-mechanics is considered. The usual "intensity" interpretation is given to the coefficients in the expansion of Ψ^n or Φ^n in terms of the eigenfunctions of an operator, although the usual probability interpretation of quantum-theory is rejected as being unsuitable for a n.-a.m.s. The locations of elementary proper masses and of elementary charges coincide only to the approximation in which (8) holds. The operators representing spin, magnetic moment, and momentum are considered in the appropriate generalisation, and the expected conclusions are drawn.

There are extremely few references. *C. Strachan.*

Wick, Gian Carlo. Sulla propagazione di un'onda di de Broglie in un mezzo materiale. (Assorbimento e polarizzazione). Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13, 1203–1233 (1942).

Rayski, George. The problem of quantization of higher order equations. Physical Rev. (2) 70, 573–574 (1946).

Coutrez, Raymond. Sur le courant quantique dans la mécanique des systèmes de corpuscules. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 250–258 (1945).

Géhéniau, J. Sur la quantification des systèmes non canoniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 214–218 (1946).

It is pointed out that the quantization of a mechanical system can be carried out even in cases where the equations of motion are not derivable from a variational function as long as the differential equations are linear. The method is to associate with the given system an adjoint system in such a way that the combined system is then derivable from a variational function. The conjugate momenta and the Hamiltonian can then be obtained and the quantization performed in the usual way. *S. Kusaka.*

De Donder, Th., et Géhéniau, J. La mécanique ondulatoire du modèle-champ. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 196–200 (1946).

This note deals with the derivation of a wave equation for a quantum field theory by a generalization of the derivation of the wave equation for a particle. Starting from a variational principle in which an element of area of a hypersurface is taken instead of an element of space-time interval, a particle differential equation associated with the variational problem is derived which is analogous to the Jacobi equation in particle mechanics. Then, by a suitable introduction of the wave function, the wave equation for a field is obtained. *S. Kusaka* (Princeton, N. J.).

De Donder, Th. La dynamique relativiste des photons. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 81–92 (1946).

Tonnelat, Marie-Antoinette. Structure générale d'une théorie unitaire des champs gravifique, électromagnétique et mésonique. C. R. Acad. Sci. Paris 222, 1162–1164 (1946). [MF 16853]

The author states without proof some formal consequences of a variational principle in which the action function is an unspecified function of a symmetric second order tensor and three (of which two are independent) anti-symmetric second order tensors. These four tensors are defined in terms of a general affine connection in a four dimensional space, and its derivatives. The connection is not assumed to be symmetric. The paper does not explain how the invariant element of volume entering into the action principle is defined. *A. H. Taub* (Seattle, Wash.).

Belinfante, F. J. On the vanishing of div $\mathfrak{E}-4\pi\rho$ in quantumelectrodynamics. Physica 12, 17–32 (1946). (English. Esperanto summary)

The present paper investigates the significance of ignorable variables in quantum mechanics, in particular, the quantum electrodynamical auxiliary condition div $\mathfrak{E}=4\pi\rho$ first discussed by Heisenberg and Pauli [Z. Physik 56, 1–61 (1929)] It is shown that these variables ($P_i=0$) impose conditions upon the domain of integration in functional space in the calculation of expectancy values and matrix elements of q-numbers. *C. Kikuchi.*

Sauter, F. Über die verschiedenen Darstellungsmöglichkeiten eines homogenen elektrischen Feldes in der Wellenmechanik. Ann. Physik (5) 43, 404–416 (1943).

The present paper considers the solutions of

$$(1)\qquad (-(\hbar^2/2\mu)\Delta-\epsilon\mathbf{r}\cdot\mathbf{F})\psi=-(\hbar/i)\partial\psi/\partial t$$

and

$$(2)\qquad (1/2\mu)((\hbar/i)\,\mathrm{grad}+\epsilon\mathbf{F}t)^2\psi=-(\hbar/i)\partial\psi/\partial t,$$

both being the wave equations of a free particle in a uniform electric field of strength **F**. Although the electric field is generally represented by a scalar potential, as in (1), it is pointed out that in the discussion of the Stark effect the wave equation with the vector potential $\epsilon\mathbf{F}t$ of the uniform field is easier to solve. It is shown by an arduous analysis that the series solution of (1) is equivalent to the solution of (2). The author also considers the perturbation calculation of $\{(1/2\mu)((\hbar/i)\ \mathrm{grad}+\epsilon\mathbf{F}t)^2+V\}\psi=-(\hbar/i)\partial\psi/\partial t$, in which the solutions of (2) are taken as the unperturbed wave functions. It is shown that the matrix elements assume simpler forms. *C. Kikuchi* (East Lansing, Mich.).

Peng, H. W. On the divergence difficulty of quantized field theories and the rigorous treatment of radiation reaction. Proc. Roy. Soc. London. Ser. A. 186, 119–147 (1946). [MF 16741]

The author sees the cause of the divergence difficulties of quantized field theories in the expansion method used hitherto, a power-series expansion with respect to the interaction constants. All terms of the resultant expansions for self-energies and cross-sections, except the term of the lowest degree, are found to diverge. The author shows that for any quantized field the system is degenerate in the sense of the perturbation theory. One has, therefore, to apply the method of secular perturbations of degenerate systems. This procedure leads here to a set of integral equations which differ from those proposed by Heitler [Proc. Cambridge Philos. Soc. 37, 291–300 (1941); these Rev 4, 95] merely by not neglecting or omitting the self-energies. The present method is to be considered as a rigorous treatment of the orthodox theory. It is applied to a simple example in which the radiation reaction is strong and leads to no divergence difficulty, whereas the usual treatment gives a divergent result. *F. W. London* (Durham, N. C.).

Eliezer, C. Jayaratnam. Radiating electron in a magnetic field. Proc. Cambridge Philos. Soc. 42, 40–44 (1946). [MF 14410]

The author's method of distinguishing between "physical" and "nonphysical" solutions of the equations of motion of Dirac's classical theory of radiating electrons [same Proc. 39, 173–180 (1943); Eliezer and Mailvaganam, same Proc. 41, 184–186 (1945); these Rev. 5, 54; 7, 101] is applied to the motion of an electron in a uniform magnetic field. The nonrelativistic approximation is solved; the exact relativistic equations are treated approximately for weak fields. The result is as expected on physical grounds. *P. Weiss* (London).

Eliezer, C. Jayaratnam. On Dirac's theory of quantum electrodynamics: the interaction of an electron and a radiation field. Proc. Roy. Soc. London. Ser. A. 187, 197–210 (1946).

Using Dirac's new formulation of quantum electrodynamics which involves the λ-limiting process and negative energy photons, the author considers the problem of the interaction of an electron with a radiation field. The wave function is developed in a series in ascending powers of the electronic charge and higher order terms are calculated which in the previous theories were divergent. It is found

that, for the elimination of all the divergences, a new condition has to be imposed; this is that only those solutions of the wave equation which correspond to outgoing waves of the electron are to be used. Then the nth order term has a nonvanishing contribution only from the part corresponding to the existence of n photons and their expressions can be written down for any n. *S. Kusaka.*

Eliezer, C. Jayaratnam. The application of quantum electrodynamics to multiple processes. Proc. Roy. Soc. London. Ser. A. **187**, 210–219 (1946).

The author's calculation of the higher order approximation terms of the interaction of an electron and a radiation field [see the preceding review] is applied to the computation of multiple processes. First a general formula is obtained for a transition probability involving one electron and three photons; from this probabilities of particular processes such as the emission of three photons by an electron, the annihilation of an electron-positron pair with the emission of three photons and the double Compton scattering are deduced. From the results it is concluded that the probabilities for multiple processes may become large if at least one of the emitted quanta is of low frequency.

S. Kusaka (Princeton, N. J.).

Costa de Beauregard, Olivier. Contribution à l'étude de la théorie de l'électron de Dirac. J. Math. Pures Appl. (9) **22**, 85–176 (1943). [MF 12310]

The author discusses the tensors constructed from a four-component spinor which is a solution of the Dirac equation. These tensors are compared with those occurring in classical relativistic theories of fluids and the electromagnetic field and analogies are discussed. Most of the material presented and the mathematical techniques used are well known.

In the first part the fundamental isomorphism between the extended Lorentz group and a subgroup of the four-dimensional complex linear group is reviewed. It is well known that this subgroup may be characterized as that group which leaves a set of four two-component spinors invariant. One of these represents an anticorrelation of period two and hence has a transformation law involving the complex conjugate of the spin image of the Lorentz transformation. The invariance of this spinor is equivalent to the condition that the Lorentz group is real in a certain coordinate system in space-time and the signature of the Hermitian matrix representing the anticorrelation is connected with the signature of the invariant quadratic form in space-time. The author states these facts in part but the clarity of statement could be greatly improved.

A. H. Taub (Seattle, Wash.).

Eriksson, H. Adolf S. On a generalization of Dirac's equations. Ark. Mat. Astr. Fys. **33B**, no. 6, 7 pp. (1946).

It is pointed out that Dirac's equation for a free electron is invariant under the Lorentz transformation which includes both spatial and temporal reflections, but that the complete Dirac-Maxwell or Dirac-Yukawa equations are invariant only for the Lorentz transformation with only spatial reflection. The author states that he has carried out the computation of a spinor representation of the orthogonal transformations in six-dimensional space and has obtained by a purely logical method generalizations of Dirac's equations which are invariant under the unrestricted Lorentz transformation. The wave function turns out to be composed of eight components and two different four-vectors can be constructed which are bilinear in the wave function though only one of them satisfies the continuity equation. Some remarks are made concerning the possibility of modifications introduced by this generalization in the theory of nuclear forces. *S. Kusaka* (Princeton, N. J.).

Beck, Guido. The physical space. Union Mat. Argentina. Memorias y Monografias (2) **1**, no. 2, 32 pp. (1944). (Spanish. English summary) [MF 16955]

In this paper the author gives an account of some of his work on the problem of constructing a geometrical interpretation of Dirac's theory of the electron. He indicates that he is still uncertain as to the final outcome of the investigation. The physical or spin space under consideration is a four-dimensional space-time continuum with a metric $ds^2=\sum_i g_i dx_i^2$ and with a further fundamental relation of the form $ds=\sum_i \gamma_i dx_i$. The γ's can be interpreted as square matrices with four rows; three of these matrices are nondiagonal. In the quantum theory nondiagonal matrices are associated with fluctuating quantities. The author concludes, therefore, that in this space "no univoque direction vector can be attributed to a given point." The properties of the space as such are not discussed further. A wave equation of the form $\sum_i \gamma_i \partial\psi/\partial x_i=(i/\Lambda)\psi$ is postulated and the greater part of the paper is devoted to a study of the transformation properties of the solutions.

L. A. MacColl (New York, N. Y.).

Beck, Guido. Sur la théorie quantique des champs statiques. I. Le champ propre d'un électron. Cahiers de Physique no. **4**, 1–4 (1941).

Pirenne, Jean. Sur la théorie quantique des champs statiques. II. L'interaction entre deux électrons. Cahiers de Physique no. **4**, 5–9 (1941).

ter Haar, D. The vibrational levels of an anharmonic oscillator. Phys. Rev. (2) **70**, 222–223 (1946).

It is shown that a rigorous solution of the Schrödinger equation for the S-states of a diatomic molecule involves the ordinary confluent hypergeometric function. Its asymptotic expression is used to check the exact formulae for the energy levels with the approximate values given by P. M. Morse [Phys. Rev. (2) **34**, 57–64 (1929)]. *N. A. Hall.*

Frenkel, J. Relativistic quantum theory of complex particles. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 326–334 (1946). (Russian. English summary)

It is shown that the classical relativistic theory is unable to describe the motion of complex particles and that it is impossible within the frame of this theory to separate the motion into the center of gravity and the relative motion, the motion of the center of gravity becoming ambiguous. These difficulties are removed if the classical theory is replaced by the quantum one. Starting from a nonrelativistic quantum description of the motion of a complex particle, it is possible to give a generalized relativistic form of the wave equation, describing the motion of the particle in such a way as if it were a material point, i.e. an elementary particle endowed with certain inner properties. This treatment is in agreement with the fact that light particles lose their individuality when they are bound together or with heavier particles if their binding energy is larger than their own rest energy. *Author's summary.*

Basu, K. Intensity calculations for the fine structure of hydrogenic atoms. I. Proc. Nat. Acad. Sci. India. Sect. A. 13, 284–300 (1943).

The author computes the relative intensities of different doublets of the Lyman series of the spectrum of hydrogen and hydrogen-like atoms. He makes use of formulas for the solution of the Dirac wave equations previously found by him, in terms of Sonine polynomials. He has to evaluate many integrals of the form

$$\int_0^\infty r^a \exp(-kr)S(br)S'(cr)dr,$$

where a, b, c, k are constants and S and S' are different Sonine polynomials. The results are compared with experimental values. *O. Frink* (State College, Pa.).

Bopp, Fritz. Lineare Theorie des Elektrons. II. Ann. Physik (5) 42, 573–608 (1942). [MF 10220]

[For part I cf. the same Ann. (5) 38, 345–384 (1940); these Rev. 2, 336.]

In Lorentz's theory of the electron the equation of motion contains, in addition to the inertia term and the radiation reaction, terms with higher derivatives of the coordinate. This means that new kinetic degrees of freedom are introduced and the trajectory of the electron cannot be defined uniquely by its initial position and velocity. Dirac has investigated the theory in which terms beyond the radiation reaction vanish and has shown that it is necessary to put a condition on the final acceleration of the electron in order to obtain sensible trajectories; thus he has had to give up the principle of causality. The present paper investigates conditions on a general theory of the electron whereby such an increase in the degree of freedom does not occur.

The first part of the paper deals with the development of a general linear theory of the electron along the lines of Mie and Born. Starting with the expression for the generalized potential the static potential, field equation and equations of motion are derived and special cases such as Dirac's theory and the author's earlier theory which combined the Maxwell field with a neutral Proca-Yukawa field are given.

The problem of the additional degrees of freedom is first considered by studying whether an electron at rest acted on by no external force can have any auxiliary motion; it is shown that it cannot if a certain general condition is satisfied. This condition, however, is not sufficient to rule out the new degrees of freedom since they may be hidden by a high degree of degeneracy which will only manifest itself in the presence of an external field. For this reason several simple types of external forces are considered and it is concluded that additional conditions must be imposed. In conclusion it is shown that the quantization of the general linear theory offers no essential difficulty. *S. Kusaka*

Hönl, H. Ist die Diracsche Theorie des Positrons Lorentzinvariant? Phys. Z. 42, 19–23 (1941).

Hönl, H. Nachtrag zu: Ist die Diracsche Theorie des Positrons Lorentzinvariant. Phys. Z. 42, 294–295 (1941).

In the first paper the author claims that many applications of the Dirac hole theory of positrons are not relativistically invariant. This claim is based on the fact that in these applications a particular frame of reference is used and ignores the fact that the results used are those shown to be independent of this choice of reference frame. It also contains a proposal for "making" the applications relativistically invariant. The second paper points out the mistake in the first and contains a simple proof of a correct result given there. *A. H. Taub* (Seattle, Wash.).

Hylleraas, Egil A. Potential walls and the so-called Klein paradox in relativistic quantum mechanics. Avh. Norske Vid. Akad. Oslo. I. 1943, no. 2, 9 pp. (1943). [MF 16436]

The Klein paradox is concerned with the mathematical possibility in the Dirac theory of the transformation of an electron to a negative energy state without emission of radiation, by the release of energy to a potential wall (that is, to a discontinuity in the potential function). Such an occurrence is physically unlikely and the effect disappears when the discontinuous potential function is replaced by a sufficiently steep continuous function. The author shows that the paradox also disappears when the potential wall is replaced by a set of potential steps. In fact, a single intermediate step is sufficient if its width is as great as the Compton wave length of the electron. *O. Frink.*

Petiau, Gérard. Sur de nouvelles relations entre densités de valeurs moyennes dans la théorie de l'électron de Dirac. Revue Sci. 83, 303–306 (1945).

Petiau, Gérard. Sur les équations d'ondes des corpuscules de spin quelconque. I. J. Phys. Radium (8) 7, 124–128 (1946).

This paper gives a short review and comparison of the wave equations which have been proposed to describe particles with arbitrary values of the spin. First de Broglie's theory is discussed in which the wave function is a product representation of Dirac's wave function for particles with spin $\frac{1}{2}$. This representation is reducible and gives rise, in addition to states of maximum spin value, to those corresponding to lower spin. Fierz's theory makes the spin value unique by eliminating the lower values with the addition of symmetry and spin conditions on the wave function. Next an "extensive theory" is described in which the wave function satisfies, not the usual second order wave equation, but an equation of higher order, and which allows the possibility of several values of the rest mass. The last theory seems to be identical with that recently proposed by Bhabha [Rev. Modern Phys. 17, 200–216 (1945); these Rev. 7, 272]. *S. Kusaka* (Princeton, N. J.).

Hjalmars, S., and Brulin, O. On the effect of cutting-off in the meson pair theory. Ark. Mat. Astr. Fys. 31B, no. 5, 7 pp. (1944).

This paper describes some methods for evaluating an improper integral arising in the Klein meson pair theory [same Ark. 30A, no. 3 (1943); these Rev. 6, 168]. The method consists of introducing a "cut-off" function in the integrand. Some results are obtained which are independent of the particular function introduced. *A. H. Taub.*

Harish-Chandra. On the scattering of scalar mesons. Proc. Indian Acad. Sci. Sect. A. 21, 135–146 (1945). [MF 12644]

The scattering of scalar mesons by a nucleon is calculated classically taking radiation damping into account. The heavy particles are assumed to have a "charge" and a "dipole." The results are compared with those of Bhabha [Proc. Roy. Soc. London. Ser. A. 172, 384–409 (1939)] for the vector meson. The results are also compared with quantum formulae for the same process which neglect radiation damping. *H. Feshbach* (Cambridge, Mass.).